An inequality index of multidimensional inequality of opportunity

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Introduction

Recent literature providing quantitative tools to measure inequality of opportunity, e.g. Checci and Peragine (2005), Lefranc et al. (2008), Ferreira and Gignoux (2008), Barros et al. (2009) (also less explicitly, Elbers et al. (2008); Lanjouw and Rao (2008))
Inequality of opportunity index

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Tools related explicitly or implicitly to different definitions of (in)equality of opportunity
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- Tools related explicitly or implicitly to different definitions of (in)equality of opportunity

- Tools much richer in the related literature of mobility but not suitable for general inequality of opportunity analysis
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- Good for handling multidimensional circumstance sets ("types") and multidimensional outcomes ("advantages")
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- Better for discrete variables (e.g. education) but applicable to continuous ones with suitable discretization
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▶ I also propose a similar index for transition matrices (in another paper)
The dissimilarity index of multidimensional inequality of opportunity: preliminaries

- Society is partitioned into "types": \( t = \{1, ..., T \} \)
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The dissimilarity index of multidimensional inequality of opportunity: preliminaries

- Society is partitioned into "types": \( t = \{1, \ldots, T\} \)
- Every type is defined by a multidimensional set of circumstances (beyond the individual’s control)
- Outcomes ("advantages") can be multidimensional. All possible combinations are in the vector: \( \alpha = \{1, \ldots, A\} \)
The dissimilarity index of multidimensional inequality of opportunity: the general family

The numerator of the index belongs to the following general family:

$$X^\beta_{T,A} \in X^* \mid X^\beta_{T,A} \equiv \sum_{t=1}^{T} \sum_{\alpha=1}^{A} N^t \frac{|p^t_{\alpha} - p^*_{\alpha}|^\beta}{p^*_{\alpha}} \quad \forall \beta, A, T \in \mathbb{N}_{++}$$
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The numerator of the index is \( X^2_{T,A} \). Also, for instance, the dissimilarity index, \( D \), of the HOI stems from \( X^1_{T,1} \).
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The numerator of the index is \( X^2_{T,A} \). Also, for instance, the dissimilarity index, \( D \), of the HOI stems from \( X^1_{T,1} \). What both applications have in common is that their maxima are easily derived.
The dissimilarity index of multidimensional inequality of opportunity itself

The index is:

\[
H \equiv \frac{X^2_{T,A}}{\text{MAX}(X^2_{T,A})} = \frac{\sum_{t=1}^{T} \sum_{\alpha=1}^{A} w^t (p_{t\alpha} - p_{\alpha}^*)^2}{\text{min}(T - 1, A - 1)}
\]

where:

- \( w^t = \frac{N^t}{N} \) and \( N = \sum_{t=1}^{T} N^t \)
Inequality of opportunity index

The dissimilarity index of multidimensional inequality of opportunity itself

The index is:

\[ H \equiv \frac{X_{T,A}^2}{\text{MAX}(X_{T,A}^2)} = \frac{\sum_{t=1}^{T} \sum_{\alpha=1}^{A} w_t (p_t^\alpha - p_*^\alpha)^2}{\text{min}(T-1, A-1)} \]

where:

- \( w^t = \frac{N_t}{N} \) and \( N = \sum_{t=1}^{T} N_t \)
- and \( p_*^\alpha = \sum_{t=1}^{T} w^t p_t^\alpha = \frac{\sum_{t=1}^{T} N_t^\alpha}{N} \)
Behaviour of the index

- The index measures inequality of opportunity understood as *dissimilarity* among conditional distributions
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Such dissimilarity is captured by the degree of association between (conditioning) types and (sets of) outcomes.
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Such dissimilarity is captured by the degree of association between (conditioning) types and (sets of) outcomes

Migration of an individual from one outcome state to another generates an *a priori* ambiguous effect because it could either increase or decrease association
Behaviour of the index

- However any migration that restores (breaks) partial equality of opportunity by pairs (e.g. equal probabilities for all type of attaining a pair of outcomes) reduces (increases) the value of the index
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- However any migration that restores (breaks) partial equality of opportunity by pairs (e.g. equal probabilities for all type of attaining a pair of outcomes) reduces (increases) the value of the index.

- Hence the index always increases when migration disturbs perfect equality of opportunity and can not increase when migration departs from a situation of perfect association.
Comparison with the criterion and index of Lefranc et al. (2008)

- For them equality of opportunity is attained when there is no second-order dominance across conditional distributions.
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- For them equality of opportunity is attained when there is no second-order dominance across conditional distributions.
- They agree with Roemer and the dissimilarity index in declaring perfect similarity as equality of opportunity.

\[ G = \frac{1}{2} \sum_{i=1}^{T} \sum_{j=1}^{T} w_i w_j |\mu_i(1-G_i) - \mu_j(1-G_j)| \]

Note that \( G = 0 \) even when the distributions are different.
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- Otherwise several distributions which they would declare as being opportunity equal are not regarded as such by the dissimilarity index (or by Roemer).
Comparison with the criterion and index of Lefranc et al. (2008)

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▶ They agree with Roemer and the dissimilarity index in declaring perfect similarity as equality of opportunity
▶ Otherwise several distributions which they would declare as being opportunity equal are not regarded as such by the dissimilarity index (or by Roemer)
▶ They also propose a Gini of inequality of opportunity:

$$G = \frac{1}{2\mu} \sum_{i=1}^{T} \sum_{j=1}^{T} w^i w^j |\mu_i (1 - G_i) - \mu_j (1 - G_j)|$$

▶ Note that $G=0$ even when the distributions are different
Example of comparison with the criterion of Lefranc et al. (2008)

The criterion of Lefranc et al. (2008) may rank both A and B as opportunity equal. However, the index ranks A as perfectly opportunity unequal and B as perfectly opportunity equal.

\[ A = \begin{pmatrix} 0 & 0.25 \\ 0.5 & 0 \\ 0 & 0.5 \\ 0.5 & 0 \\ 0 & 0.25 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \]
Comparison with the types approach of Checci and Peragine (2005) and Ferreira and Gignoux (2008)

- They use path-independent, decomposable indices (usually the mean log deviation)
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- The types approach measures inequality by comparing a standard of the distribution (typically the mean) across groups/types
- By contrast, the dissimilarity index measures inequality in terms of a distance which is related to the degree of association between types and outcomes
- They propose both relative measures (between-group inequality divided by total inequality) and absolute ones (between-group inequality)
Continuing the comparison

- This different criteria leads to at least two instances of discrepancy
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1. The types approach may rank society A as opportunity equal (if row values are 1, 2, 3, 4, 5)
This different criteria leads to at least two instances of discrepancy

1. The types approach may rank society A as opportunity equal (if row values are 1,2,3,4,5)

2. When two societies exhibit maximum association the index ranks both as opportunity unequal but the relative version of the types-approach measurement is sensitive to within-group inequality, hence it will rank one society above another (in the version without the ELMO adjustment)
Example

The index ranks C and D as perfectly opportunity unequal but the relative type approach ranks C as exhibiting less inequality of opportunity.

\[ C = \begin{pmatrix} 
0 & 0 \\
0.5 & 0 \\
0 & 1 \\
0 & 0 \\
0.5 & 0 
\end{pmatrix} \quad D = \begin{pmatrix} 
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0 \\
1 & 0 
\end{pmatrix} \]
Comparison with the tranches approach of Checci and Peragine (2005)

- Again path-independent decomposable indices are used (e.g. mean log deviation)
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- Again path-independent decomposable indices are used (e.g. mean log deviation)
- They divide the percentile space into tranches and replace each observation with the mean value corresponding to the specific tranche-type cell
- Inequality of opportunity is measured as within-tranch inequality
Comparison with the tranches approach of Checci and Peragine (2005)

- As with the type approach, there is an absolute and a relative version of the index (within tranch divided by total inequality)
Comparison with the tranches approach of Checci and Peragine (2005)

- As with the type approach, there is an absolute and a relative version of the index (within tranch divided by total inequality)
- The tranches approach agrees with the dissimilarity index in declaring perfect equality of opportunity if and only if conditional distributions are identical
Inequality of opportunity index
Comparison with other concepts and indices

Comparison with the tranches approach of Checci and Peragine (2005)

- As with the type approach, there is an absolute and a relative version of the index (within tranch divided by total inequality)
- The tranches approach agrees with the dissimilarity index in declaring perfect equality of opportunity if and only if conditional distributions are identical
- However the relative version of the tranches approach may rank differently two societies exhibiting perfect association of types and outcomes if and when there is differential between-tranches inequality
Example

The index ranks E and F as perfectly opportunity unequal but the relative tranches approach ranks E as exhibiting less (relative) inequality of opportunity.

\[ E = \begin{pmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{pmatrix} \]

\[ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \]
Comparison with the inequality index, D, of the HOI; Barros et al. (2009)

The HOI considers both the average attainment of an outcome of interest and its dispersion across groups (following, among others, Atkinson (1970), Sen (1976), Yitzhaki (1979)). The dissimilarity index is only concerned with inequality of opportunities.
Comparison with the inequality index, $D$, of the HOI; Barros et al. (2009)

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- Its index capturing inequality is:

$$D \equiv \frac{X_{T,1}^1}{2N} = \frac{1}{2} \sum_{t=1}^{T} w^t \frac{|p^t_1 - p^*_1|}{p^*_1}$$
Comparison with the inequality index, D, of the HOI; Barros et al. (2009) continued

- D works for dichotomous variables (e.g. access to a service) whereas H can be used for both dichotomous and more general multinomial cases
Comparison with the inequality index, D, of the HOI; Barros et al. (2009) continued

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Comparison with the inequality index, D, of the HOI; Barros et al. (2009) continued

- D works for dichotomous variables (e.g. access to a service) whereas H can be used for both dichotomous and more general multinomial cases
- The normalization of D is not appealing when dealing with $A \geq 2$, it depends ad hoc on group weights
- So it is interesting to observe the differences between H and D when applied to similar cases (e.g. access to a service)
Comparison with the inequality index, D, of the HOI; Barros et al. (2009) continued

- D and $H_{T,2}^2$ agree on declaring inequality of opportunity if and only if conditional distributions are equal across types.
Comparison with the inequality index, D, of the HOI; Barros et al. (2009) continued

- D and $H^2_{T,2}$ agree on declaring inequality of opportunity if and only if conditional distributions are equal across types.
- However they do not agree always when declaring *perfect* inequality of opportunity.
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- For D perfect inequality exists when one individual (within one type) has full access and everybody else has no access whatsoever.
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- Whereas for H perfect inequality exists in all different situations where there is perfect association between outcomes and types
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For D perfect inequality exists when one individual (within one type) has full access and everybody else has not access whatsoever.

Whereas for H perfect inequality exists in all different situations where there is perfect association between outcomes and types.

Hence whenever D declares perfect inequality H follows but the opposite is not true.
Comparison with the inequality index, D, of the HOI; Barros et al. (2009) continued

- Plus they do not always agree in ranking pairs of societies in intermediate cases of imperfect inequality
Comparison with the inequality index, D, of the HOI; Barros et al. (2009) continued

- Plus they do not always agree in ranking pairs of societies in intermediate cases of imperfect inequality.
- Hence how to decide among them in circumstances when such decision may be relevant (i.e. dichotomous variables)?
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- Plus they do not always agree in ranking pairs of societies in intermediate cases of imperfect inequality
- Hence how to decide among them in circumstances when such decision may be relevant (i.e. dichotomous variables)?
- My suggestion: use D for access problems (e.g. to sanitation, health facilities, etc.) and use $H_{T,2}^2$ for dichotomous outcomes whenever there is no a priori hierarchy between the two outcome options (e.g. blue collar vs. white collar)
Comparison with the inequality index, D, of the HOI; Barros et al. (2009) continued

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- Hence how to decide among them in circumstances when such decision may be relevant (i.e. dichotomous variables)?
- My suggestion: use D for access problems (e.g. to sanitation, health facilities, etc.) and use $H^2_{T,2}$ for dichotomous outcomes whenever there is no a priori hierarchy between the two outcome options (e.g. blue collar vs. white collar).
- Even when the use of D is conceptually justified, D never hits its maximum (like the Gini index; an implicit trade-off between normalization and population invariance, which is absent in the case of $H^2_{T,2}$).
Empirical application

▶ Comparison of inequality of opportunity in Peru
▶ Cohort of adults 45+ versus 22-45
▶ Did inequality increase or decrease?
Empirical application: data

- Outcomes: years of education (4 categories) and quality of education (3 categories: no, public, private)
- Circumstances: Gender, father’s and mother’s education (each 2 categories: up to complete primary, more than complete primary)
- Data: Peruvian National Household Survey, ENAHO 2001
Empirical application: results

Dissimilarity index for two cohorts of Peruvian adults

<table>
<thead>
<tr>
<th>Cohort</th>
<th>45+</th>
<th>22-45</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point estimate</td>
<td>0.044698</td>
<td>0.039811</td>
</tr>
<tr>
<td>Lower confidence interval</td>
<td>0.040367</td>
<td>0.037614</td>
</tr>
<tr>
<td>Upper confidence interval</td>
<td>0.05012</td>
<td>0.042935</td>
</tr>
</tbody>
</table>

99% confidence intervals
Concluding remarks

- The dissimilarity index measures inequality of opportunity in proportion to the degree of to which circumstance sets associate with outcome sets.
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Inequality of opportunity index

Concluding remarks

The comparison with other indices highlights that:

1. All indices agree to classify societies as opportunity equal when conditional distributions of well-being are identical.
2. Several indices or conceptions declare equality of opportunity even when conditional distributions are not identical.
3. The dissimilarity index (with the tranches approach) declares equality of opportunity if and only if conditional distributions are identical.
4. Different notions of what inequality means explain discrepancies in classification of unequal societies among indices (even among those which claim influence by Roemer's notion).
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- Advantage of the index: well-suited to deal with multidimensional outcomes
- Potential disadvantage: it requires discretization of continuous variables
- Empirical application: The index is helpful to show a small but statistically significant in reduction of inequality of opportunity among a younger cohort of Peruvians (measured in terms of the circumstances and outcomes considered)