Convergence Forward and Backward?\textsuperscript{1}

Quentin Wodon and Shlomo Yitzhaki*  
World Bank and Hebrew University  

March 2005

Abstract

This note clarifies the relationship between $\beta$-convergence and $\sigma$-convergence in a univariate setting. In such a setting, $\sigma$-convergence implies $\beta$-convergence, but $\beta$-convergence does not imply $\sigma$-convergence. The results imply that caution should be used for the interpretation of the very concept of $\beta$-convergence in empirical work.

Key words: convergence, growth

JEL categories: O11, O40, D63

1. Introduction

The concepts of $\sigma$-convergence and $\beta$-convergence have been used extensively in the literature. $\sigma$-convergence is said to occur when the dispersion in per capita GDP between countries or regions falls

\textsuperscript{1} The views expressed here are those of the authors and need not reflect those of The World Bank. We thank Yevgeny Artzev, Danny Quah, Michael Beenstock, David Johnson, Branko Milanovic, David Weil and participants of seminars at Hebrew University, Tel-Aviv University and the Bank of Israel for their comments on a previous version of this paper. We thank the anonymous referee for the comments that helped to improve the paper.

* Corresponding author:
Address: Shlomo Yitzhaki  
Dept. of Economics  
Hebrew University  
Jerusalem, 91905  
Israel  
E-Mail: shlomo.yitzhaki@huji.ac.il  
Phone: +972-2-6592201  
Fax: +972-2-6522319
over time. $\beta$-convergence, a concept emerging from neo-classical growth models assuming diminishing returns in production, refers to a potentially negative relationship between growth in per capita GDP and the initial level of income of a country, so that poorer countries may grow faster than richer countries, and thereby catch up with these richer countries, controlling for other characteristics.

A number of studies have aimed to test empirically whether $\beta$-convergence has been observed. While initial studies suggested a rate of convergence at about 2 percent per year (e.g., Barro, 1991; Barro and Sala-i-Martin, 1992; Sala-I-Martin, 1996), more recent research has put these initial findings in doubt (e.g., Quah, 1993, 1996; Friedman, 1992; Caselli et al., 1996; Bliss, 1999; Cannon and Duck, 2000). There have also been empirical studies of $\sigma$-convergence, with the results depending in part on the choice of the measure of dispersion or inequality used (e.g., Dalgaard and Vastrup, 2001).

In a recent paper that we became aware of at the time the present note was accepted for publication, Furceri (2005) shows that $\sigma$-convergence implies $\beta$-convergence but that is only a necessary condition for $\sigma$-convergence. In this note, we go further in using a simple univariate setting to discuss the relationship between $\beta$-convergence and $\sigma$-convergence, and to show a puzzling result, namely that $\beta$-convergence can be observed both forward and backward in time! That is, we may observe that the regression coefficient of second period income as a function of initial income, and the regression coefficient of initial income as a function of second period income are both smaller than one. A sufficient condition for two-way $\beta$-convergence in our univariate setting is that there be no $\sigma$-convergence or $\sigma$-divergence over time, which may happen quite easily (an empirical example is given). While our result does not imply that conditional $\beta$-convergence can also be observed both forwards and backward (this would depend on the data set at hand and the multivariate specification used), the simple possibility of such a finding suggests that caution should be used for the interpretation of the very concept of $\beta$-convergence in empirical work.
2. The relationship between $\beta$-convergence and $\sigma$-convergence

Consider two periods in time, and a log transformation $y = \ln(Y)$, where $Y$ is real income per capita. Define $\sigma_i (i=1,2)$ as the standard deviation of $y_i$, while $\beta_{ij}$ is the regression coefficient of $y_i$ on $y_j$. $\sigma$-convergence implies that $\sigma$ declines over time, while $\beta$-convergence occurs when $\beta_{21} < 1$, that is the regression coefficient of $y_2$ on $y_1$ is less than 1. To simplify the discussion, we restrict ourselves to simple univariate regressions. Equation (1) gives the relationship between the two concepts:

$$\frac{\sigma^2_2}{\sigma^2_1} = \frac{\text{cov}(y_2, y_2)}{\text{cov}(y_1, y_1)} = \frac{\text{cov}(y_1, y_1)}{\text{cov}(y_2, y_1)} = \frac{\text{cov}(y_2, y_1)}{\text{cov}(y_2, y_2)} = \frac{\beta_{21}}{\beta_{12}} . \tag{1}$$

$\sigma$-convergence means $\sigma_2 < \sigma_1$, so that the left hand-side of (1) is less than one. Note, however, by definition:

$$\beta_{12} \beta_{21} = \frac{\text{cov}(y_1, y_2) \text{cov}(y_2, y_1)}{\text{cov}(y_2, y_2) \text{cov}(y_1, y_1)} = \rho^2 \leq 1 , \tag{2}$$

where $\rho$ is the Pearson correlation coefficient between $y_1$ and $y_2$. Using (2) we can write:

$$\beta_{21} = \frac{\rho^2}{\beta_{12}} . \tag{3}$$

Using (3), one can rewrite (1) as:

$$\frac{\sigma^2_2}{\sigma^2_1} = \frac{\beta_{21}}{\beta_{12}} = \frac{\beta^2_{21}}{\rho^2} . \tag{4}$$

If there is $\sigma$-convergence, i.e., the left hand side of (4) is less than one, then it must be that there is $\beta$-convergence. On the other hand, $\beta$-convergence does not necessarily imply $\sigma$-convergence, because $\rho < 1$. A decline in inequality implies $\beta < 1$, but $\beta$ can be less than one even if inequality is rising, provided that there is sufficient rank switching (i.e., low correlation) between countries. In the extreme

---

2 We are only dealing here with concepts and population parameters. No assumption is made on whether the model is linear, nor on the data generating process. See Bliss (1999) and Cannon and Duck (2000) for a discussion of the econometric issues.
case where the distributions $y_1$ and $y_2$ are statistically independent, then $\beta=0$ whether inequality is rising or declining.\(^3\)

Equation (2) also indicates that we may face an index number problem because both $\beta_{21}$ and $\beta_{12}$ can be lower than one, which could be interpreted as convergence whether one moves forward or backward in time. Actually, a sufficient condition for two-way $\beta$-convergence is no change in $\sigma$. Indeed, $\beta_{12} < 1$ implies that $\sigma_{12} < \sigma_2^2$, where $\sigma_{12}$ is the covariance between the incomes in the two periods. Similarly $\beta_{21} < 1$ implies that $\sigma_{12} < \sigma_1^2$. Hence, $\sigma_{12} < \text{Min} [\sigma_1^2, \sigma_2^2]$ is the condition that both betas are lower than one. $\sigma_1^2 = \sigma_2^2$ and $\rho < 1$ insure that this condition holds.

It is not difficult in practice to find two-way $\beta$-convergence empirically, for example using Penn6 data for Latin American countries. Denoting as before $y=\ln(Y)$, we estimated that $y_{1998} = 2.26 + 0.79 \ y_{1960}$, and $y_{1960} = 3.17 + 0.57 \ y_{1998}$, with both slopes being statistically significant at the 5 percent level. This type of problems should raise doubts with respect to any conclusions one may draw from $\beta$ being less than one\(^4\).

It is also important to realize that the problem with the empirics of $\beta$-convergence is also present when the Gini index is used as an alternative to the variance of the logarithm of per capita GDP.\(^5\) The equivalent of equation (2) in a Gini framework is:

$$
\beta_{12}^G \beta_{21}^G = \frac{\text{cov}(Y_1, F_2) \ 	ext{cov}(Y_1, F_2)}{\text{cov}(Y_2, F_2) \ 	ext{cov}(Y_1, F_1)} = \Gamma_{21} \Gamma_{12} \leq 1,
$$

where $F_1$ and $F_2$ are a country’s normalized ranks (zero for the poorest country and one for the richest country) in the world distribution of per capita income, $\beta_{12}^G$ and $\beta_{21}^G$ are the Gini regression coefficients of income in one period versus the other period (Olkin and Yitzhaki, 1992), and $\Gamma_{21}$ and $\Gamma_{12}$ are the Gini

---

\(^3\) This point can be contrasted with the dominant view in the literature as captured from the following quote: “Put in another way, $\beta$-convergence is a necessary condition for $\sigma$-convergence” (Sala-i-Martin, 1996, p-1329.) See Sala-i-Martin (2002) for a recent review of the literature.

\(^4\) The possibility of simultaneous backwards and forwards convergence is mentioned in Quah (1993), p. 432, but not developed formally, nor tested empirically.

\(^5\) For an analysis of convergence using the Gini framework, see Wodon and Yitzhaki (2005).
correlations between periods 2 and 1, and between periods 1 and 2 (Schechtman and Yitzhaki, 1987). As is the case for the Pearson correlation coefficient, the Gini correlations are bounded by minus one (perfect reranking, i.e. the richest country in period one becomes the poorest in period two, and so on) and one (no reranking at all, i.e. inequality can decrease or increase, but the countries keep their original rank). In general, $\Gamma_{21}$ need not be equal to $\Gamma_{12}$. Similar to OLS, we can write the Gini regression coefficient $\beta_{21}^G$ as:

$$
\beta_{21}^G = \frac{\text{cov}(Y_2, F_1)}{\text{cov}(Y_1, F_1)} = \frac{\text{cov}(Y_2, F_1) \cdot \text{cov}(Y_2, F_2)}{\text{cov}(Y_2, F_2) \cdot \text{cov}(Y_1, F_1)} = \Gamma_{21}^G \frac{G_2 \mu_2}{G_1 \mu_1}
$$

(8)

where $G_1$ and $G_2$ are the Gini indices of inequality in periods 1 and 2, and convergence implies that $G_2 < G_1$. Since the Gini correlation coefficient $\Gamma_{21}$ is smaller or equal to one, $\beta_{21}^G$-convergence may occur even if there is no Gini convergence. On the other hand, if Gini convergence occurs, then $\beta_{21}^G$-convergence must also occurs. If the Gini correlations are equal to one, i.e., if there is no reranking at all between the two periods, Gini convergence is identical to $\beta_{21}^G$-convergence (abstracting from changes in mean income). If reranking occurs between the two periods, then $\beta_{21}^G$-convergence can occur even if there is no decrease in inequality. For example, if the observations in the two periods are statistically independent, then $\beta_{21}^G$ is equal to zero independently of what happens to the Gini index of inequality, and the convergence is observed forward and backward, as for the OLS case.

3. Conclusion

The aim of this note was to clarify the relationship between $\beta$-convergence and $\sigma$-convergence in a simple univariate setting. We have argued that in such a setting, if there is $\sigma$-convergence, then there

---

6 One advantage of the Gini framework over the variance of the logarithm is that under the Gini estimation procedure, one does not have to correct for non-linearity caused by different growth rates (non-linearity can be a source of a bias in OLS estimation using logarithms, as noted in footnote 3, p. 1356, in Quah, 1996).
must be $\beta$-convergence, but $\beta$-convergence does not imply $\sigma$-convergence (a result independently shown as well by Furceri, 2005). In addition, we have argued that $\beta$-convergence may be observed both forward and backward. While we have focused here on a univariate setting, it would be useful to test for the possibility of bi-directional $\beta$-convergence in a multivariate setting as well, using conditional convergence methodologies.
References


Friedman, M., 1992, Do old fallacies ever die?, *Journal of Economic Literature* 30, 2129-2132.


