The dynamics of asset prices and leverage under belief heterogeneity*

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Abstract
The recent economic crisis highlights the role of financial markets in allowing economic agents, including prominent banks, to speculate on the future returns of different financial assets, such as mortgage-backed securities. This paper introduces a dynamic general equilibrium model with aggregate shocks, endogenously incomplete markets and heterogeneous agents to investigate this role of financial markets. In addition to their risk aversion and endowments, agents differ in their beliefs about the future exogenous states (aggregate and idiosyncratic) of the economy. This difference in beliefs induces them to take large bets under frictionless complete financial markets, which enable agents to leverage their future wealth. Consequently, as hypothesized by Friedman (1953), under complete markets, agents with incorrect beliefs will eventually be driven out of the markets. In this case, they also have no influence on asset prices in the long run. In contrast, I show that under incomplete markets generated by collateral constraints, agents with heterogeneous (potentially incorrect) beliefs survive in the long run and the movement in the financial wealth distribution between agents with different beliefs permanently drive up asset price volatility. The movement in the wealth distribution also generates various patterns of booms and busts in asset prices observed in Burnside, Eichenbaum, and Rebelo (2011). Lastly, I use this framework to study the effects of financial regulation and of the financial wealth distribution on leverage and asset price volatility.

1 Introduction
The events leading to the financial crisis of 2007-2008 have highlighted the importance of belief heterogeneity and how financial markets create opportunities for agents with different

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beliefs to leverage up and speculate. Several investment and commercial banks invested heavily in mortgage-backed securities, which subsequently suffered large declines in value. At the same time, some hedge funds profited from the securities by short-selling them.

One reason for why there has been relatively little attention, in economic theory, paid to heterogeneity of beliefs and how these interact with financial markets is the *market selection hypothesis*. The hypothesis, originally formulated by Friedman (1953), claims that in the long run, there should be limited differences in beliefs because agents with incorrect beliefs will be taken advantage of and eventually be driven out the markets by those with the correct belief. Therefore, agents with incorrect beliefs will have no influence on economic activity in the long run. This hypothesis has recently been formalized and extended in recent work by Blume and Easley (2006) and Sandroni (2000). However these papers assume financial markets are complete and this assumption plays a central role in allowing agents to pledge all their wealth.

In this paper, I present a dynamic general equilibrium framework in which agents differ in their beliefs but markets are endogenously incomplete because of collateral constraints. Collateral constraints limit the extent to which agents can pledge their future wealth and ensure that agents with incorrect beliefs never lose so much as to be driven out of the market. Consequently, all agents, regardless of their beliefs, survive in the long run and continue to trade on the basis of their heterogeneous beliefs. This leads to additional asset price volatility (relative to a model with homogeneous beliefs or relative to the limit of the complete markets economy).

The dynamic general equilibrium approach adopted here is central for the investigation of survival and disappearance of agents as well as their effect on asset prices. Since the approach permits the use of well-specified collateral constraints, it enables me to look at whether agents with incorrect beliefs will be eventually driven out of the market. It also allows for a comprehensive study of leverage and a characterization of the effects of financial regulation on economic fluctuations.\(^1\)

More specifically, I study an economy in dynamic general equilibrium with both aggregate shocks and idiosyncratic shocks and heterogeneous, infinitely-lived agents.\(^2\) The shocks follow a Markov process. Consumers differ in terms of their beliefs on the transition matrix of the Markov process (for simplicity, these belief differences are never updated as there is no learning; in other words agents in this economy agree to disagree).\(^3\) There is a unique final

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\(^1\)In Cao (2010), I show that the dynamic stochastic general equilibrium model with endogenously incomplete markets presented here is not only useful for the analysis of the effects of heterogeneity in the survival of agents with different beliefs, but also includes well-known models as special cases, including recent models, such as those in Fostel and Geanakoplos (2008) and Geanakoplos (2009), as well as more classic models including those in Kiyotaki and Moore (1997) and Krusell and Smith (1998). For instance, a direct generalization of the current model allows for capital accumulation with adjustment costs in the same model in Krusell and Smith (1998) and shows the existence of a recursive equilibrium. The generality is useful in making this framework eventually applicable to a range of questions on the interaction between financial markets, heterogeneity, aggregate capital accumulation and aggregate activity.

\(^2\)Infinite horizon and infinitely-lived agents allow the use of stationary equilibria and the analysis of short run versus long run.

\(^3\)Alternatively, one could assume that even though agents differ with respect to their initial beliefs, they partially update them. In this case, similar results would apply provided that the learning process is sufficiently slow (which will be the case when individuals start with relatively firm priors). In the paper, I
consumption good, and several real and financial assets. The real assets, which I sometimes refer to as *Lucas trees* as in Lucas (1978), are in fixed supply. I assume that agents cannot short sell these real assets.

Endogenously incomplete (financial) markets are introduced by assuming that all loans have to use financial assets as collateralized promises as in Geanakoplos and Zame (2002). Selling a financial asset is equivalent to borrowing and in this case, agents need to put up some real assets as collateral. Loans are non-recourse and there is no penalty for defaulting. Consequently, whenever the face value of the security is higher than the value of its collateral, the seller of the security can choose to default without further consequences. In this case, the security buyer seizes the collateral instead of receiving the face value of the security. I refer to equilibria of the economy with these financial assets as *collateral constrained equilibria*\(^4,5\). Several key results involve the comparison of collateral constrained equilibria to the standard competitive equilibrium with complete markets.

Households (consumers) can differ in many aspects, such as risk-aversion and endowments. Most importantly, they differ in their beliefs concerning the transition matrix governing transitions across the exogenous states of the economy. Given the consumers’ subjective expectations, they choose their consumption and real and financial asset holdings to maximize their intertemporal expected utility.

The framework delivers several results. The first set of results, already mentioned above, is related to the survival of agents with incorrect beliefs. As in Blume and Easley (2006) and Sandroni (2000), with perfect, complete markets, in the long run, only agents with correct beliefs survive. Their consumption is bounded from below by a strictly positive number. Agents with incorrect beliefs see their consumption go to zero, as uncertainties realize. However, in any collateral constrained equilibrium, every agent survives because of the constraints. When agents lose their bets, they can simply walk away from their collateral while keeping their current and future endowments to come back and trade in the financial markets in the same period.\(^6\) They cannot do so under complete markets because they can commit to delivering all their future endowments.\(^7\)

More importantly, the survival or disappearance of agents with incorrect beliefs affects asset price volatility. Under complete markets, agents with incorrect beliefs will eventually be driven out of the markets in the long run. The economies converge to economies with homogeneous beliefs, i.e., the correct belief. Markets completeness then implies that asset prices in these economies are independent of the past realizations of aggregate shocks. In

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\(^4\)I avoid using the term *incomplete markets equilibria* to avoid confusion with economies with missing markets. Markets can be complete in the sense of having complete spanning set of financial assets. But the presence of collateral constraints introduces *endogenously incomplete* markets because not all positions in these financial assets can be taken.

\(^5\)Collateral constrained equilibria are closer to liquidity constrained equilibria than to debt-constrained equilibria in Kehoe and Levine (2001), in which the authors show that the dynamics of the former is much more complex than the one of the latter. Liquidity constrained economies are special cases of collateral constrained economies when the set of financial assets is chosen to be empty.

\(^6\)The collateral constraints is a special case of limited commitment because if there will be no need for collateral if agents can fully commit to their promises.

\(^7\)Even though the survival mechanism here is relatively simple (but also realistic), characterizing equilibrium variable such as asset prices and leverage in this environment is not an easy exercise.
addition, asset prices are the net present discounted values of the dividend processes with appropriate discount factors. As a result, asset price volatility is proportional to the volatility of dividends if the aggregate endowment, or equivalently the equilibrium stochastic discount factor, only varies by a limited amount over time and across states. These properties no longer hold in collateral constrained economies. Given that agents with incorrect beliefs survive in the long run, they exert permanent influence on asset prices. Asset prices are not only determined by the aggregate shocks as in the complete markets case, but also by the evolution of the wealth distribution across agents. This also implies that asset prices are history-dependent as the realizations of past aggregate shocks affect the current wealth distribution. The additional dependence on the wealth distribution raises asset price volatility under collateral constraints above the volatility level under complete markets.

I establish this result more formally using a special case in which the aggregate endowment is constant and the dividend processes are I.I.D. Under complete markets, asset prices are asymptotically constant. Asset price volatility, therefore, goes to zero in the long run. In contrast, asset price volatility stays well above zero under collateral constraints as the wealth distribution changes constantly, and asset price depends on the wealth distribution. Although this example is extreme, numerical simulations show that its insight carries over to less special cases. In general, long-run asset price volatility is higher under collateral constraints than under complete markets.

The volatility comparison is different in the short run, however. Depending on the distribution of endowments, short run asset price volatility can be greater or smaller under complete or under collateral constraints. This result happens because the wealth distribution matters for asset prices under both complete markets and under collateral constraints in the short run. This formulation also helps clarify the long-run volatility comparison. In the long run, under complete markets, the wealth distribution becomes degenerate as it concentrates only on agents with correct belief. In contrast, under collateral constraints, the wealth distribution remains non-degenerate in the long run and affects asset price volatility permanently. However, the wealth of agents with incorrect beliefs may remain low as they tend to lose their bets. Strikingly, under collateral constraints and when the set of actively traded financial assets is endogenous, the poorer the agents with incorrect beliefs are, the more they leverage to buy assets. High leverage generates large fluctuations in their wealth, and as a consequence, large fluctuations in asset prices.\(^8\)

It is also useful to highlight the role of dynamic general equilibrium for some results mentioned above. In particular, the infinite horizon nature of the framework allows a comprehensive analysis of short-run and long-run behavior of asset price volatility. Such an analysis is not possible in finite horizon economies, including Geanakoplos’s important study on the effects of heterogeneous beliefs on leverage and crises. For example, in page 35 of Geanakoplos (2009), he observes similar volatility as the economy moves from collateral constrained economies to complete markets economies. In my model, the first set of results described above shows that the similarity holds only in the short run. The long run dynamics of asset price volatility totally differs from complete markets to collateral constrained

\(^8\)Similarly, in Cao (2010) I show that the results concerning volatility of asset prices also translate into volatility of physical investment, i.e., capital accumulation. Physical investment under collateral constraints and heterogeneous beliefs exhibits higher volatility than under complete markets.
economies. In my model, the results are also based on insights in Blume and Easley (2006) and Sandroni (2000) regarding the disappearance of agents with incorrect beliefs. However, these authors do not focus on the effect of their disappearance on asset price or asset price volatility. Proposition 4 in this paper also strengthens the result on asset prices in Sandroni (2000).

The dynamic general equilibrium of the economy also captures the "debt-deflation" channel as in Mendoza (2010), which models a small open economy. The economy in my paper also follows two different dynamics in different times, "normal business cycles" and "debt-deflation cycles," depending on whether the collateral constraints are binding for any of the agents. In a debt-deflation cycle, the collateral constraint binds. Then, when a bad shock hits the economy, the constrained agents are forced to liquidate their physical asset holdings. This fire sale of the physical assets reduces the price of these assets and tightens the constraints further and starting a vicious circle of falling asset prices. This paper shows that the debt-deflation channel still operates when we are in a closed-economy with endogenous interest rate, as opposed to exogenous interest rates as in Mendoza (2010). Moreover, due to this mechanism, asset price volatility also tends to be higher at low levels of asset price near the debt-deflation region. This pattern has been documented in several empirical studies, including Heathcote and Perri (2011). The movement in wealth distribution also generates the patterns of booms and busts observed in Burnside, Eichenbaum, and Rebelo (2011).

The second set of results that follows from this framework concerns collateral shortages. I show that collateral constraints will eventually be binding for every agent in any collateral constrained equilibrium provided that the face values of the financial assets with collateral span the complete set of state-contingent Arrow-Debreu securities, i.e., markets are complete in the spanning sense but endogenously incomplete due to collateral constraints. Intuitively, if this was not the case, as proved for complete markets, the unconstrained asset holdings would imply arbitrarily low levels of consumption at some state of the world for every agent, contradicting the result that consumption is bounded from below. In other words, there are always shortages of collateral. This result sharply contrasts with those obtained when agents have homogenous beliefs but still have reasons to trade due to differences in endowments or utility functions. In these cases, if the economy has enough collateral, then collateral constraints may not bind and the complete markets allocation is achieved. Heterogeneous beliefs, therefore, guarantee collateral shortages, but not other dimensions of heterogeneity, such as heterogeneity in risk-aversion or endowment.

Another immediate implication of these results concerns Pareto inefficiency of collateral constrained equilibria. Collateral constrained equilibria are Pareto-suboptimal whenever agents strictly differ in their beliefs. This can be seen for the results that under complete markets equilibria, some agent’s consumption will come arbitrarily close to zero while this never happens under collateral constraints. Intuitively, under complete markets agents pledged their future income, while collateral constraints put limits on such transactions. While allocations in which some agents experience very low levels of consumption may not be attractive according to some social welfare criteria, the equilibrium under complete markets is Pareto optimal under the subjective expectations of the agents. This result also implies that there is the possibility for Pareto improving regulations. However, given that this result is about unconstrained Pareto-efficiency, Pareto improving regulations might in-
olve altering the collateral constraints.\footnote{For a two-period version of my model, the concept of constrained Pareto-inefficient due to Geanakoplos and Polemarchakis (1986) can be checked. In some cases, the economy can be constrained inefficient in this sense, due to pecuniary externalities.}

The above mentioned results are derived under the presumption that collateral constrained equilibria exist. However, establishing existence of collateral constrained equilibria is generally a challenging task. The third set of results establishes the existence of collateral constrained equilibria with a stationary structure. I look for Markov equilibrium, i.e., in which equilibrium prices and quantities depend only on the distribution of normalized financial wealth. I show the existence of the equilibria under standard assumptions. I also develop an algorithm, to compute these equilibria. A similar algorithm can be used to compute the complete markets equilibrium benchmark.

The fourth set of results attempts to answer some normative questions in this framework. Simple and extreme forms of financial regulations such as shutting down financial markets are not beneficial. Using the algorithm described above, I provide numerical results illustrating that these regulations fail to reduce asset price volatility and moreover they may also reduce the welfare of all agents because of the restrictions they impose on mutually beneficial trades. The intuition is for the greater volatility under such regulations is similar to the intuition for why long run asset price volatility is higher under collateral constrained economies than under complete markets economies explained earlier. Financial regulations act as further constraints protecting the agents with incorrect beliefs. Thus in the long run these agents hold most of the assets which they believe, incorrectly, to have high rates of returns. The shocks to the rates of returns on these assets then create large movements in the marginal utilities of the agents, hence large volatility of the prices of the assets. These results regarding regulation suggest that Pareto-improving or volatility reducing regulations must be sophisticated, for example, incorporating state-dependent regulations.

This paper is related to the growing literature studying collateral constraints, started with a series of papers by John Geanakoplos. The dynamic analysis of collateral constrained equilibria is related to Kubler and Schmedders (2003). They pioneer the introduction of financial markets with collateral constraints into a dynamic general equilibrium model with aggregate shocks and heterogeneous agents. The technical contribution of this paper relative to Kubler and Schmedders (2003) is to introduce heterogeneous beliefs using Radner (1972) rational expectations equilibrium concept: even though agents assign different probabilities to both aggregate and idiosyncratic shocks, they agree on the equilibrium outcomes, including prices and quantities, once a shock is realized. This rational expectations concept differs from the standard rational expectation concept, such as the one used in Lucas and Prescott (1971), in which subjective probabilities should coincide with the true conditional probabilities given all the available information.\footnote{Another technical contribution in Cao (2010), is to introduce capital accumulation and production in a tractable way. Capital accumulation or physical investment is modelled through intermediate asset producers with convex adjustment costs that convert old units of assets into new units of assets using final good. Lorenzoni and Walentin (2009) models capital accumulation with adjustment cost using used capital markets. Through asset producers, I assume markets for both used and new capital.}

Related to the survival of agents with incorrect beliefs, Coury and Sciuoba (2005) and Beker and Chattopadhyay (2009) suggest a mechanism for agents’ survival based on explicit
debt constraints as in Magill and Quinzii (1994). These authors do not consider the effects of the agents’ survival on asset prices. My framework is tractable enough for a simultaneous analysis of survival and its effects on asset prices. As mentioned in footnote 6, collateral constraints are a special case of limited commitment. However, this special case of limited commitment is in contrast to the usual limited commitment literature where agents are assumed to be banned from trading in financial markets after their defaults such as in Kehoe and Levine (1993) and Alvarez and Jermann (2001). In this paper, agents can always come back to the financial markets and trade starting with their endowment after defaulting and loosing all their financial wealth. Given this better outside option, the financial constraints are more stringent then they are in the other papers. Beker and Espino (2010) has a similar survival mechanism to mine based on the limited commitment framework in these papers. However, my approach to asset pricing is different because asset prices are computed explicitly as a function of wealth distribution. Moreover, my approach also allows a comprehensive study of asset-specific leverage. Kogan, Ross, Wang, and Westerfield (2006) and Borovicka (2010) explore yet another survival mechanism based on the preferences of agents but use complete markets instead.

My paper is also related to the literature on the effect of heterogeneous beliefs on asset prices studied in Xiong and Yan (2009) and Cogley and Sargent (2008). These authors, however, consider only complete markets. The survival of irrational traders is studied Long, Shleifer, Summers, and Waldmann (1990) and Long, Shleifer, Summers, and Waldmann (1991) but they do not have a fully dynamic framework to study the long run survival of the traders. Simsek (2009b) also studies the effects of belief heterogeneity on asset prices, but in a static setting. He assumes exogenous wealth distributions to investigate the question whether heterogeneous beliefs affect asset prices. In contrast, I study the effects of the endogenous wealth distribution on asset prices as well as asset price volatility. Simsek (2009a) focuses on consumption volatility. He shows that as markets become more complete, consumption becomes more volatile as agents can speculate more. My first set of results suggests that this comparative statics only holds in the short run. In the long run, the reverse statement holds due to market selection.

The channel through which asset prices deviate from their fundamental values is different from the limited arbitrage story in Shleifer and Vishny (1997). In their story, the deviation arises because agents with correct beliefs hit their financial constraints before being able to arbitrage away the price anomalies. In this paper, agents with incorrect beliefs hit their financial constraint more often and are protected by the constraint. Moreover, in the equilibria computed in Section 5.2, agents with the correct belief (the pessimists) never hit their borrowing constraint.

When capital accumulation is introduced, in Cao (2010), the model presented here is a generalization of Krusell and Smith (1998) with financial markets and adjustment costs. In particular, the existence theorem 2 shows that a recursive equilibrium in Krusell and Smith (1998) exists. Krusell and Smith (1998) derives numerically such an equilibrium, but they do not formally show its existence. My paper is also related to Kiyotaki and Moore (1997), although I provide a microfoundation for the financial constraint (3) in their paper using the

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endogeneity of the set of actively traded financial assets.

At the time of the first draft of this paper in 2009, I was not aware of the recent papers that discuss some issues related to the ones I consider in this paper. Brumm, Grill, Kubler, and Schmedders (2011) show the importance of collateral requirements on asset price volatility in a similar model but with two trees and Epstein-Zin recursive preferences. Kubler and Schmedders (2011) show the importance of beliefs heterogeneity and wealth distribution on asset prices in a model with overlapping-generations.

The rest of the paper proceeds as follow. In Section 2, I present the general model of an endowment economy and preliminary analysis of survival, asset price volatility under the complete markets benchmark as well as under collateral constraints. In Section 3, I define and show the existence of collateral constrained equilibria under the form of Markov equilibria. In this section, I also prove important properties of Markov equilibria in this model. In Section 4, I derive a general numerical algorithm to compute Markov and competitive equilibria. Section 5 focuses on assets in fixed supply with an example of only one asset to illustrate the ideas in Sections 2 and 3. In Cao (2010), I present the most general model with capital accumulation, labor supply and production. Section 6 concludes with potential applications of the framework in this paper. Lengthy proofs and constructions are in Appendices A and B and in Cao (2010).

2 General model

In this general model, there are heterogeneous agents who differ in their beliefs about the future streams of dividends. There are also different types of assets (for examples trees, land, housing and machines) that differ in their dividend process and their collateral value. For example, some of the assets can be used as collateral to borrow and others cannot. These assets are in fixed supplied as in Lucas (1978) in order to study the effects of belief heterogeneity on asset prices. In Cao (2010), I show that the model can also allow for assets in flexible supply and production in order to study the effects of belief heterogeneity on aggregate physical investment and aggregate economy activity. Assets in fixed supply presented in this paper are special cases of assets in flexible supply with adjustment costs approaching infinity.

2.1 The endowment economy

Consider an endowment, a single consumption (final) good economy in infinite horizon with infinitely-lived agents (consumers). Time runs from $t = 0$ to $\infty$. There are $H$ types of consumers

$$h \in \mathcal{H} = \{1, 2, \ldots, H\}$$

in the economy with a continuum of measure 1 of identical consumers in each type. These consumers might differ in many dimensions including per period utility function $U_h(c)$ (i.e., risk-aversion), discount rates $\beta_h$, and endowments of good $e_h$. The consumers might also differ in their initial endowment of real assets, Lucas’ trees,\(^{12}\) that pay off real dividends in

\(^{12}\)See Lucas (1978)
terms of the consumption good. However, the most important dimension of heterogeneity is
the heterogeneity in beliefs over the evolution of the exogenous state of the economy. There
are $S$ possible exogenous states (or equivalently shocks)

$$s \in S = \{1, 2, \ldots, S\}.$$  

The states capture both idiosyncratic uncertainties, i.e., individual endowments, and aggre-
gate uncertainties, i.e., the dividends from the physical assets.$^{13}$

The evolution of the economy is captured by the past and current realizations of the
shocks over time: $s^t = (s_0, s_1, \ldots, s_t)$ is the series of realizations of shocks up to time $t$.
Notice that the space $S$ can be chosen large enough to encompass both aggregate shocks,
such as shocks to the aggregate dividends, and idiosyncratic shocks, such as individual
endowment shocks. I assume that the shocks follow a Markov process with the transition
probabilities $\pi(s, s')$. In order to rule out transient states, I make the following assumption.

**Assumption 1** $S$ is ergodic.

Now, in contrast to the standard rational expectation literature, I assume that the agents
do not have a perfect estimate of the transition matrix $\pi$. Each of them has their own estimate
of the matrix, $\pi^h$. However, these estimates are not very far from the truth, i.e., there exist
$u$ and $U$ strictly positive such that

$$u < \frac{\pi^h(s, s')}{\pi(s, s')} < U \quad \forall s, s' \in S \text{ and } h \in \mathcal{H}$$  

where $\pi(s, s') = 0$ if and only if $\pi^h(s, s') = 0$ in which case let $\frac{\pi^h(s, s')}{\pi(s, s')} = 1$. This formulation
allows for time varying belief heterogeneity as in He and Xiong (2011). In particular, agents
might share the same beliefs in good states, $\pi^h(s, .) = \pi^{h'}(s, .)$, but their beliefs start
diverging in bad states, $\pi^h(s, .) \neq \pi^{h'}(s, .)$. $^{15}$ Notice that (1) implies that every agent
believes that $S$ is ergodic.

**Real Assets:** As mentioned above, there are $A$ real assets $a \in A = \{1, 2, \ldots, A\}$. These
assets pay off state-dependent dividends $d_a(s)$ in final good. These assets can both be
purchased and be used as collateral to borrow. This gives rise to the notion of leverage on
each asset. The ex-dividend price of each unit of asset $a$ in history $s^t$ is denoted by $q_a(s^t)$.
I assume that agents cannot short-sell these real assets.$^{16}$ The total supply $K_a$ of asset $a$ is
given at the beginning of the economy, under the form of asset endowments to the consumers.

$^{13}$A state $s$ can be a vector $s = (A, e_1, \ldots, e_H)$ where $A$ consists of aggregate shocks and $e_h$ are idiosyncratic
shocks.

$^{14}$Learning can be easily incorporated into this framework as into this framework by allowing additional
state variables which are the current beliefs of agents in the economy. As in Blume and Easley (2006) and
Sandroni (2000), agents who learn slower will disappear under complete markets. However they all survive
under collateral constraints. The dynamics of asset prices describe here will correspond to the short-run
behavior of asset prices in the economy with learning.

$^{15}$Simsek (2009b) shows in a static model that only the divergence in beliefs about bad states matters for
asset prices.

$^{16}$I can relax this assumption by allowing limited short-shelling.
Financial Assets: In each history \(s^t\), there are also (collateralized) financial assets, \(j \in J\). Each financial asset \(j\) (or financial security) is characterized by a pair of vectors, promised payoffs and collateral requirement \((b_j, k_j)\). Promises are a standard feature of financial asset similarly to Arrow’s securities, i.e., asset \(j\) traded in history \(s^t\) promises next-period pay-off \(b_j(s^{t+1}) = b_j(s_{t+1}) > 0\) in term of final good at the successor nodes \(s^{t+1} = (s^t, s_{t+1})\). The non-standard feature is the collateral requirement. Agents can only sell the financial asset \(j\) if they hold shares of real assets as collateral. We associate \(j\) with an \(A\)-dimensional vector \(k^j = (k_a^j)_{a \in A}\) of collateral requirements. If an agent sells one unit of security \(j\), she is required to hold a portfolio \(k_a^j \geq 0\) units of asset for each \(a \in A\), as collateral.\(^{17}\)

Since there are no penalties for default, a seller of the financial asset defaults at a node \(s_{t+1}\) whenever the total value of collateral assets falls below the promise at that state. By individual rationality, the actual pay-off of security \(j\) at node \(s^t\) is therefore always given by

\[
f_{j,t+1}(s^{t+1}) = \min \left\{ b_j(s_{t+1}), \sum_{a \in A} k_a^j (q_a(s^{t+1}) + d_a(s^{t+1})) \right\}. \quad (2)
\]

Let \(p_{j,t}(s^t)\) denote price of security \(j\) at node \(s^t\).

Assumption 2 Each financial asset requires at least a strictly positive collateral

\[
\min_{j \in J} \max_{a \in A} k_a^j > 0
\]

If a financial asset \(j\) requires no collateral then its effective pay-off, determined by (2) will be zero, it will be easy to show that in equilibrium its price, \(p_j\), will be zero as well. We can thus ignore these financial assets.

Remark 1 The financial markets incomplete endogenously even if \(J\) is complete in the usual sense of complete spanning, i.e., \(J \geq S\). Because agents are constrained in the positions they can take due to the collateral requirement and the fact that the total supply of collateral assets is finite. The collateral requirement is a special case of limited commitment, as if borrowers (sellers of the financial assets) have full commitment ability, they will not be required to put up any collateral to borrow.\(^{18}\)

Remark 2 Consider the case in which a financial asset \(j\) requires only \(k_a^j\) units of asset \(a\) as collateral. Selling one unit of financial asset \(j\) is equivalent to purchasing \(k_a^j\) units of asset \(a\) and at the same time pledging these units as collateral to borrow \(p_{j,t}\). It is shown in Cao (2010) that

\[k_a^j q_a,t - p_{j,t} > 0,\]

that is the seller of the financial asset always has to pay some margin. So the decision to sell the financial asset \(j\) using the real asset \(a\) as collateral corresponds to the desire to invest into asset \(a\) at margin rather than the simple desire to borrow. In Cao and Gete (2011), we allow for the possibility of under-collateralization, thus the ability to pledge some asset to borrow, over and above investing at margin motive in this paper.

\(\text{\(^{17}\)Notice that, there are only one-period ahead financial assets. See He and Xiong (2011) for a motivation why longer term collateralized financial assets are not used in equilibrium.}\)

\(\text{\(^{18}\)Alvarez and Jermann (2000) is another example of asset pricing under limited commitment.}\)
Remark 3 We can then define the leverage ratio on asset \(a\) associated with the transaction as

\[
L_{j,t} = \frac{k^j_{a,t} q_{a,t}}{k^j_{a,t} q_{a,t} - p_{j,t}} = \frac{q_{a,t} - \frac{p_{j,t}}{k^j_{a,t}}}{q_{a,t}}. \tag{3}
\]

Even though there are many financial assets available, in equilibrium only some financial asset will be actively traded, which in turn determines which leverage levels prevail in the economy. In this sense, both asset price and leverage are simultaneously determined in equilibrium, as emphasized in Geanakoplos (2009).

Remark 4 It is shown in Cao (2010) that the set \(J\) of financial securities can be assumed dependent on the history of the economy \(s^t\), provided that there exists a \(k > 0\) such that

\[
\inf_{j \in J} \max_{a \in A} k^j_a > k.
\]

For example, \(J_t\) may contain a financial security \(j\) that requires only an asset \(a\) as collateral with the collateral requirement depending on the history \(s^t\) and

\[
k^j_{a,t} = \max_{s^{t+1} | s^t} \left\{ \frac{b_j (s_{t+1})}{q_a (s^{t+1}) + d_a (s^{t+1})} \right\}. \tag{4}
\]

So in this example \(k^j_{a,t}\) is the minimum collateral level that ensures no default, that is

\[
f_{j,t+1} (s^{t+1}) = b_j (s_{t+1}) \forall s_{t+1} \in S.
\]

This constraint captures the situation in Kiyotaki and Moore (1997) in which agents can borrow only up to the minimum across future states of the future value of their land. With \(S = 2\), and state non-contingent debts, i.e., \(b_j (s_{t+1}) = b_j\), Geanakoplos (2009) argues that even if we allow for a wide range of collateral level, that is the unique collateral level that prevails in equilibrium (thus there is a unique level of leverage in each instance, according to the remark above). This statement for two future states still holds in this context of infinitely-lived agents as proved later in Section 5. However, this might not be true if we have more than two future states.

Consumers: Consumers are the most important actors in this economy. They can be hedge fund managers or banks’ traders in financial markets. In each state \(s^t\), each consumer is endowed with a potentially state dependent endowment \(e^h_t = e^h (s_t)\) units of the consumption good. I suppose there is a strictly positive lower bound on these endowments. This lower bound guarantees a lower bound on consumption if a consumer decides to default on all her debt.

Assumption 3 \(\min_{h,s} e_h (s) > \xi > 0\).

An example of this assumption is that commercial banks receive deposits from their retail branches while these banks also have trading desks that trade independently in the financial markets.

Consumers maximize their intertemporal expected utility with the per period utility functions \(U_h (\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}\) that satisfy
Assumption 4 \( U_h \) is concave and strictly increasing.

Notice that I do not require \( U_h \) to be strictly concave. This assumption captures linear utility functions in Geanakoplos (2009) and Harrison and Kreps (1978).

Consumer \( h \) takes the sequences of prices \( \{q_{a,t}, p_{j,t}\} \) as given and solves

\[
\max_{\{c_t^h, k_{a,t}^h, \phi_{j,t}^h\}} \mathbb{E}^h \left[ \sum_{t=0}^{\infty} \beta^t U_h \left( c_t^h \right) \right] 
\]

and in each history \( s^t \), she is subject to the budget constraint

\[
c_t^h + \sum_{a \in A} q_{a,t} k_{a,t}^h + \sum_{j \in J} p_{j,t} \phi_{j,t}^h \leq c_t^h + \sum_{j \in J} f_{j,t} \phi_{j,t-1}^h + \sum_{a \in A} (q_{a,t} + d_{a,t}) k_{a,t-1}^h 
\]

the collateral constraints

\[
k_{a,t}^h + \sum_{j: \phi_{j,t}<0} \phi_{j,t}^h k_{a,t}^j \geq 0 \forall a \in A 
\]

One implicit condition from the assumption on utility functions is that consumptions are positive, i.e., \( c_t^h \geq 0 \). In the constraint (7), if the consumer does not use asset \( a \) as collateral to sell any financial security, then the constraint becomes the no-short sale constraint

\[
k_{a,t}^h \geq 0
\]

The most important feature of the objective function is the superscript \( h \) in the expectation operator, \( \mathbb{E}^h \), that represents the subjective beliefs when agents estimate their future expected utility. The expectation can also be re-written explicitly as \( \sum_{t,s^t} P_h (s^t) \beta^t U_h \left( c_t^h (s^t) \right) \), where \( P_h (s^t) \) is the probability of history \( s^t \) under agent \( h \)'s belief. Entering period \( t \), agent \( h \) holds \( k_{a,t-1}^h \) old units of real asset \( a \) and \( \phi_{j,t-1}^h \) units of financial asset \( j \). She can trade old units of real asset \( a \) at price \( q_{a,t} \) and buy new units of asset \( k_{a,t}^h \) for time \( t+1 \) at the same price. She can also buy and sell financial securities \( \phi_{j,t}^h \) at price \( p_{j,t} \). If she sells financial securities she is subject to collateral requirement (7).

At first sight, the collateral constraint (7) does not have the usual property of financial constraints in the sense that higher asset prices do not seem to enable more borrowing. However, using the definition of the effective pay-off, \( f_{j,t} \), in (2), we can see that this effective pay-off is increasing in the prices of physical assets, \( q_{a,t+1} \). As a result, financial asset prices, \( p_{j,t} \), are also increasing in physical asset prices. So borrowers can borrow more if \( q_{a,t+1} \)'s increase. Given that the borrowing constraints is effective through future asset prices, when we embed this channel in a production economy in Cao (2010), this constraint creates a feedback mechanism from the financial sector to the real sector similar to Kiyotaki and Moore (1997).

Even though the formulation and solution method presented below allow for heterogeneity in the discount rates, to focus on beliefs heterogeneity, I assume in from now on that agents have the same discount factor.

---

19 I also introduce the disutility of labor in the general existence proof in Cao (2010) in order to study employment in this environment. The existence of equilibria for finite horizon allows for labor choice decision. When we have strictly positive labor endowments, \( l^h \), we can relax Assumption 3 on final-good endowments, \( e^h \).

20 This channel is different from the one in Brunnermeier and Sannikov (2010).
Assumption 5  Agents have the same discount factor $\beta_h = \beta \ \forall h \in \mathcal{H}$.

Equilibrium: In this environment, I define an equilibrium as follows

**Definition 1** An **collateral constrained equilibrium** for an economy with initial asset holdings

$$\{k_{a,0}\}_{h \in \{1,2,\ldots,H\}}$$

and initial shock $s_0$ is a collection of consumption, real and financial asset holdings and prices in each history $s^t$,

$$\{\begin{array}{l}
\{c_t^h(s^t), k_{a,t}^h(s^t), \phi_{j,t}^h(s^t)\}_{h \in \{1,2,\ldots,H\}} \\
\{q_{a,t}(s^t)\}_{a \in A}, \{p_{j,t}(s^t)\}_{j \in J_t(s^t)}
\end{array}$$

satisfying the following conditions:

i) Asset markets for each real asset $a$ and for each financial asset $j$ in each period clear:

$$\sum_{h \in \mathcal{H}} k_{a,t}^h(s^t) = K_a$$

$$\sum_{h \in \mathcal{H}} \phi_{j,t}^h(s^t) = 0.$$  

ii) For each consumer $h$, $\{c_t^h(s^t), k_{a,t}^h(s^t), \phi_{j,t}^h(s^t)\}$ solves the individual maximization problem subject to the budget constraint, (6), and the collateral constraint, (7).

Notice that by setting the set of financial securities $J$ empty, we obtain a model with no financial markets in which agents are only allowed to trade in real assets, but they cannot short-sell these assets. This case corresponds to Lucas (1978)’s model with several trees and heterogeneous agents.

As a benchmark, I also study equilibria under complete financial markets. Consumers can borrow and lend freely by buying and selling Arrow-Debreu state contingent securities, only subject to the no-Ponzi condition.\(^{21}\) In each node $s^t$, there are $S$ financial securities. Financial security $s$ delivers one unit of final good if state $s$ happens at time $t+1$ and zero units otherwise. Let $p_{s,t}$ denote time $t$ price and let $\phi_{s,t}^h$ denote consumer $h$’s holding of this security. The budget constraint (6) of consumer $h$ becomes

$$c_t^h + \sum_{a \in A} q_{a,t} k_{a,t}^h + \sum_{s \in S} p_{s,t} \phi_{s,t}^h \leq c_t^h + \phi_{s,t}^h + \sum_{a \in A} (q_{a,t} + d_{a,t}) k_{a,t-1}^h$$  

**Definition 2** A **complete markets equilibrium** is defined similarly to an incomplete markets equilibrium except that each consumer solves her individual maximization problem subject to the budget constraint (9) and the no-Ponzi condition, instead of the collateral constraints (7).

In the next subsection, I establish some properties of collateral constrained markets equilibrium. I compare each of these properties to the one of complete markets equilibrium.

\(^{21}\)No-Ponzi condition

$$\lim_{t \to \infty} \sum_{s \in S} \prod_{r=0}^{t-1} p_{s,r+1}(s^r) \phi_{s,t}^h \geq 0.$$
2.2 General properties of collateral constrained and complete markets equilibria

Given the endowment economy, we can easily show that total supply of final good in each period is bounded by a constant $\bar{e}$. Indeed in each period, total supply of final good is bounded by

$$\bar{e} = \max_{s \in S} \left( \sum_{h \in H} e^h(s) + \sum_{a \in A} d_a(s) K_a \right)$$

(10)

The first term of the right hand side is the total endowment of each individual. The second term is total dividends from the real assets. In collateral constrained or complete markets equilibria, the market clearing condition for final good implies that total consumption is bounded from above by $\bar{e}$. Given that consumption of every agent is always positive, consumption of each agent is bounded from above by $\bar{e}$, i.e.,

$$c_{h,t}(s') \leq \bar{e} \forall t, s'. $$

(11)

Under the fixed supply of real assets, we can show that in any collateral constrained equilibrium, the consumption of each consumer is bounded from below by a strictly positive constant $c$. Two assumptions are important for this result. First, no-default-penalty allows consumers, at any moment in time, to walk away from their past debts and only lose their collateral assets. After defaulting, they can always keep their non-financial wealth (inequality (13) below). Second, increasingly large speculation by postponing current consumption is not an equilibrium strategy, because in equilibrium, consumption is bounded by $\bar{e}$, in inequality (14). This assumption prevents agents from constantly postponing their consumption to speculating in the real assets and is main difference with the survival channel in Alvarez and Jermann (2000) used by Beker and Espino (2010) for heterogeneous beliefs which will be explained in detail below. Formally, we have the following theorem.

**Theorem 1** Suppose that there exists a $c$ such that

$$U_h(c) < \frac{1}{1 - \beta} U_h(\bar{e}) - \beta \frac{1}{1 - \beta} U_h(\bar{e}), \forall h \in H,$$

(12)

where $\bar{e}$ is defined in (10). Then in a collateral constrained equilibrium, consumption of each consumer in each history always exceeds $c$.

**Proof.** This result is shown in an environment with homogenous beliefs (Lemma 3.1 in Duffie, Geanakoplos, Mas-Colell, and McLennan (1994)). It can be done in the same way under heterogenous beliefs. However, I replicate the proof in order to provide the economic intuition in this environment. As in (11), we can find an upper bound for consumption of each consumer in each future period. Also, in each period, one of the feasible strategies of consumer $h$ is to default on all her past debts at the only cost of losing all the collateral.
assets, but she can still at least consume her endowment from the current period on, therefore
\[
U_h(c_{h,t}) + \mathbb{E}_t^h \left[ \sum_{r=1}^{\infty} \beta^r U_h(c_{h,t+r}) \right] \geq \frac{1}{1 - \beta} U_h(e) \tag{13}
\]

Notice that in equilibrium, \( \sum_h c_{h,t+r} \leq \bar{e} \) therefore \( c_{h,t+r} \leq \bar{e} \). So
\[
U_h(c_h) + \frac{\beta}{1 - \beta} U_h(\bar{e}) \geq \frac{1}{1 - \beta} U_h(e)
\tag{14}
\]

This implies
\[
U_h(c_h) \geq \frac{1}{1 - \beta} U_h(e) - \frac{\beta}{1 - \beta} U_h(\bar{e}) > U_h(e).
\]

Thus, \( c_h \geq e \).

Condition (12) is satisfied immediately if \( \lim_{c \to 0} U_h(c) = -\infty \), for example, with log utility or CRRA utility with CRRA constant exceeding 1.

The survival mechanism here is similar to the one in Alvarez and Jermann (2000) and Beker and Espino (2010). In particular, the first term in the right hand side of (12) captures the fact that the agents always have the option to default and go to autarky in which they only consume their endowment which exceeds \( e \) in each period, which is the lower bound for consumption in Alvarez and Jermann (2000) and Beker and Espino (2010). However, the two survival mechanisms also differ because, in this paper, agents can always default on their promises and lose all their physical asset holdings, but they can always go back to financial markets to trade right after defaulting. The second term in the right hand side of (12) captures the fact that, this possibility might hurt the agents if they have incorrect beliefs. The prospect of higher reward for speculation, i.e. high \( \bar{e} \), will induce these agents to constantly postponing consumption to speculate. As a result, their consumption level might fall well below \( e \). Indeed, the lower bound of consumption, \( e \), is decreasing in \( \bar{e} \): the more there is of the total available final good, the more profitable speculative activities are and the more incentives consumers have to defer current consumption to engage in these activities.

One immediate corollary of Proposition 1 is that every consumer survives in equilibrium. Therefore, collateral constrained equilibrium differs from complete markets equilibrium when consumers differ in their beliefs. The proposition below shows that in a complete markets equilibrium, with strict difference in beliefs, consumption of certain consumers will come arbitrarily close to 0 at some history. The intuition for this result is that if an agent believes that the likelihood of a state is much smaller than what other agents believe, the agent will want to exchange his consumption in that state for consumption in other states. Complete markets allow her to do so but collateral constraints limit the amount of consumption that she can sell in each state.

**Proposition 1** Suppose there are consumers with the correct belief and some consumers with incorrect beliefs. Moreover, the utility functions satisfy the Inada-condition
\[
\lim_{c \to 0} U_h''(c) = +\infty \quad \forall h \in \mathcal{H}.
\tag{15}
\]
Then, in a complete markets equilibrium, almost surely

$$\lim_{t \to \infty} c^h_t = 0$$  \hspace{1cm} (16)

for each $h$ such that $\pi^h$ differs from $\pi$, that is $h$ has incorrect belief.

**Proof.** In Appendix A, I show that the conditions in Proposition 5 in Sandroni (2000) are satisfied. Thus the proposition implies that almost surely, for agents $h$ with incorrect beliefs, we have

$$\lim_{t \to \infty} \frac{P_h(s^t)}{P(s^t)} = 0.$$  \hspace{1cm} (16)

(16) is a direct consequence given the first-order conditions on the holding of Arrow-Debreu securities:

$$\frac{U'_h(c_h(s^t))}{U'_{h'}(c_{h'}(s^t))} = \frac{P_{h'}(s^t)U'_h(c_h(s_0))}{P_h(s^t)U'_{h'}(c_{h'}(s_0))} \forall h, h' \in \mathcal{H},$$  \hspace{1cm} (17)

together with the facts that there are agents with correct beliefs, i.e. $P_{h'} = P$, the Inada condition (15), and bounded consumption (11) applied for $c_{h'}$.

**Corollary 1** In a complete markets equilibrium, under the Inada condition (15), if agents strictly differ in their beliefs, then consumption of some agent approaches zero at some history of the world. Formally,

$$\inf_{h,s^t} c_h(s^t) = 0.$$  \hspace{1cm} (18)

**Proof.** If agents strictly differ in their beliefs, there exists a pair of agents $h, h'$ such that $\pi^h \neq \pi^{h'}$. Applying Proposition 1 for $P = P_{h'}$ we have, under $P_{h'}$, (16) happens almost surely. This directly implies (18).

Notice that this result is rather surprising because even if agents are strictly risk-averse, they can also disappear over time if they have incorrect beliefs. Because they can perfectly commit to pay their creditor using their future income. They can do so using short-term debts and keep rolling over their debts while using their present income to pay interest, which grows over time as their indebtedness grows. This result also sheds light on the survival mechanism in Theorem 1: agents have limited ability to pledge their future income, for example their labor income, to their creditors. As a result, they can always default and keep their future income. This limited commitment is even stronger in my setting than in Alvarez and Jermann (2000) and Beker and Espino (2010) because after defaulting, agents can always come back and trade in the financial markets by buying new physical assets and then use these physical assets to sell financial assets. This story is also different from the

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22 If the consumers with incorrect beliefs are risk-neutral, their consumption will go to zero immediately after a certain date.

23 In Chien and Lustig (2009), in equilibrium, borrowers are indifferent between defaulting and not defaulting due to the complete spanning properties of financial assets. In my model, when there is not complete spanning, there will be strict defaults in equilibrium. The numerical method in Section 4 can be used to solve for equilibria in Chien and Lustig (2009) as well. See Hopenhayn and Werning (2008) for a model in which equilibrium defaults happen due to stochastic outside options.

24 The following story of the founder of Long Term Capital Management shows that traders in the financial markets
limited arbitrage story in Shleifer and Vishny (1997), where asset prices differ from their fundamental values because agents with correct beliefs hit their financial constraints before being able to arbitrage away the price difference. Here, agents with incorrect beliefs hit their financial constraint more often and are protected by the constraint. Moreover, in the equilibria computed in Section 5.2, agents with the correct belief (the pessimists) never hit their borrowing constraint.

Due to different conclusions about agents’ survival, the following corollary asserts that complete markets and collateral constrained allocations strictly differ when some agents strictly differ in their beliefs.

**Corollary 2** Suppose that conditions in Theorem 1 and Proposition 1 are satisfied. Then, a collateral constrained equilibrium never yields an allocation that can be supported by a complete markets equilibrium. By the Second Welfare Theorem, collateral constrained equilibrium allocations are Pareto-inefficient.

**Proof.** In a collateral constrained equilibrium, consumptions are bounded away from 0, but in a complete markets equilibrium, consumptions of some agents will approach 0. Therefore, the two sets of allocations never intersect. ■

Using this corollary, we can formalize and show the shortages of collateral assets.

**Proposition 2 (Collateral Shortages)** If financial markets are complete in terms of spanning, i.e., the set of the vectors of promises $b_j$ has full rank. Then, for any given time $t$, with positive probability, the collateral constraints must be binding for some agent after time $t$.

**Proof.** We prove this corollary by contradiction. Suppose none of the collateral constraints are binding after a certain date. Then we can take the first-order condition with respect to the state-contingent securities. This leads to consumption of some agents to approach zero at infinity, as shown in the proof of Proposition 1. This contradicts the conclusion of Theorem 1 that consumption of each agent is bounded away from zero. ■

Notice that, as in Lucas (1978), agents can hold the real assets for the risk-return and consumption-investment trade-offs. However, when their collateral constraints are binding, the agents use these assets solely as collateral to borrow. I interpret the binding collateral constraints as collateral shortages. Araujo, Kubler, and Schommer (2009) argue that when there is enough collateral we might reach the Pareto optimal allocation. However, in the complete markets case, there will never be enough collateral.\(^{25}\)


\(^{25}\)I show in Cao (2010) that even if the supply of these collateral assets is endogeneous, there will still be collateral shortages.
I also emphasize here the difference between belief heterogeneity and other forms of heterogeneity such as heterogeneity in endowments or in risk-aversion. The following proposition, in the same form as Theorem 5 in Geanakoplos and Zame (2007), shows that if consumers share the same belief and discount rate, there exist endowment profiles with which collateral equilibria attain the first-best allocations.

**Proposition 3** If consumers share the same belief and discount factor, there is an open set of endowment profiles with the properties that the competitive equilibrium can be supported by a collateral constrained equilibrium.

**Proof.** We start with an allocation such that there is no trade in the complete markets equilibrium, then as we move to a neighborhood of that allocation, all trade can be collateralized.

Even though other dimensions of heterogeneity such as risk-aversion and endowments also create trading in financial markets, this proposition shows the importance of belief heterogeneity in driving up trading volume and resulting in binding collateral constraints.

Before moving to show the existence and study the properties of collateral constrained equilibria, we go back to the complete markets benchmark to study the behavior of asset price volatility. We will compare this volatility with volatility under collateral constraints and show that, in general, in the long run, asset price is more volatile in a collateral constrained equilibrium than it is in a complete markets equilibrium.

**Proposition 4** Suppose that there are some agents with the correct belief, in the complete markets equilibrium, almost surely asset prices converge to the prices prevailing in an economy in which there are only agents with the correct belief. In particular, the prices are independent of the past realizations of the aggregate shocks, as they are functions of only the current aggregate shock. Formally, there exists a set of asset prices $\bar{q}_a (s)$ as functions of the aggregate state of the economy such that, almost surely, for any sequence of history $\{s^t\}$:

$$\lim_{t \to \infty} \left\{ \sup_{r \geq 0} \left| q_a \left( s^{t+r} \right) - \bar{q}_a (s_{t+r}) \right| \right\} = 0. \quad (19)$$

**Proof.** The detailed proof is in Appendix A. Proposition 1 shows that in the long run, only agents with the correct belief survive. Therefore, in the long run, the economy converges to the economy with homogeneous belief (rational expectation). In such an economy, given markets completeness, there exists a representative agent with an instantaneous utility function $U_{\text{Rep}}$, and her marginal utility evaluated at the total endowment determines asset prices

$$\bar{q}_a \left( s^t \right) U'_{\text{Rep}} \left( e(s_t) \right) = \beta \mathbb{E}_t \left\{ \left( \bar{q}_a \left( s^{t+1} \right) + d_a (s_{t+1}) \right) U'_{\text{Rep}} \left( e(s_{t+1}) \right) \right\}$$

$$= \mathbb{E}_t \left\{ \sum_{r=1}^{\infty} d_a (s_{t+r}) \beta^r U'_{\text{Rep}} \left( e(s_{t+r}) \right) \right\} \quad (20)$$

in which $e(s)$ is the aggregate endowment in the aggregate state $s$. We can see easily from this expression that $\bar{q}_a \left( s^t \right)$ is history-independent.
Under complete markets, asset price does depend on the endogenous wealth distribution and the exogenous state \( s \), as shown in the example below. However, in the long run, as the economy converges to an homogenous beliefs economy, the wealth distribution converges to a limiting distribution. So in the long run and under complete markets, asset prices only depend on the aggregate state. In the short run, as the wealth distribution also moves over time, asset price volatility might be very large.

**Remark 5** Proposition (4) is stronger than Proposition 6 in Sandroni (2000) in two ways. First, the functions \( \bar{\pi}_a (s) \) are independent of history path. Second, the convergence in (19) is strong in the sense that the sup is taken over \( r \geq 0 \) instead of any finite number as in Sandroni (2000).

To illustrate Proposition 4, in Appendix A, I derive the following closed form solution of asset price under complete markets

**Example 1** In the case of log utility, the equilibrium price of assets is a weighted sum of the prices

\[
q_a (s^t) = \sum_{h \in \mathcal{H}} \hat{\omega}_h (s^t) \left\{ \sum_{r=0}^{\infty} \sum_{s^{t+r} \mid s^t} \beta^r P_h (s^{t+r} \mid s^t) d (s_t) \frac{e (s_t)}{e (s_{t+r})} \right\},
\]

where

\[
\hat{\omega}_h (s^t) = \frac{w_h (s^t)}{\sum_{h' \in \mathcal{H}} w_{h'} (s^t)};
\]

and

\[
w_h (s^t) = \sum_{r=0}^{\infty} \sum_{s^{t+r} \mid s^t} p (s^{t+r} \mid s^t) c_h (s^{t+r}).
\]

In the long run, as agents \( h \) with incorrect beliefs disappear in the limit, i.e., \( \lim_{t \to \infty} w_h (s^t) = 0 \), or equivalently \( \lim_{t \to \infty} \hat{\omega}_h (s^t) = 0 \), all the wealth in the economy concentrates on agents with correct beliefs, i.e., \( \lim_{t \to \infty} \sum_{h \in \mathcal{H}, P_h = P} \hat{\omega}_h (s^t) = 1 \) so \( q_a (s^t) \) approaches

\[
\bar{\pi}_a (s^t) = \sum_{r=0}^{\infty} \sum_{s^{t+r} \mid s^t} \beta^r P_h (s^{t+r} \mid s^t) d (s_t) \frac{e (s_t)}{e (s_{t+r})},
\]

which depends only on \( s_t \) due to the Markov property of the evolution of the exogenous state.

A direct corollary of Proposition 4 is the following:

**Corollary 3** When the dividend process of an asset is I.I.D. and the aggregate endowment is constant across states, with probability one, the price of the asset converges to a constant in the long run.

**Proof.** By assumption \( e (s_{t+r}) = e \) for all \( t \) and \( r \). So \( U'_{\text{rep}} (e) \) cancels out in both sides of (20). As a result

\[
\bar{\pi}_a (s^t) = E_t \left\{ \sum_{r=1}^{\infty} d_a (s_{t+r}) \beta^r \right\} = \frac{\beta}{1 - \beta} \sum_{s \in S} \pi (s) d_a (s).
\]
The second equality comes from the fact that shocks are I.I.D.

In contrast to complete markets equilibrium, in the next section I show that, in a collateral constrained equilibrium, asset prices can be history-dependent, as past realizations of aggregate shocks always affect the wealth distribution, which in turn affects asset prices.

One issue that might arise when one tries to interpret Proposition 4 is that, in some economy, there might not be any consumer whose belief coincides with the truth. For example, in Scheinkman and Xiong (2003), all agents can be wrong all the time, except they constantly switch from over-optimistic to over-pessimistic.\textsuperscript{26} To avoid this issue, I again use the language in Blume and Easley (2006) and Sandroni (2000). I reformulate the results above using the subjective belief of each consumer.

**Proposition 5** Suppose that the Inada condition (15) is satisfied. Then each agent believes that:

1) In complete markets equilibrium, only her and consumers sharing her belief survive in the long run. However, in collateral constrained equilibrium, everyone survives in the long run.

2) In complete markets equilibrium, asset prices are history-independent. However, in collateral constrained equilibrium, asset price can be history-dependent.

The properties in this section are established under the presumption that collateral constrained equilibria exist. The next section is devoted to show the existence of these equilibria with a stationary structure. Then Section 4 presents an algorithm to compute the equilibria.

## 3 Markov Equilibrium

In this section, I show that collateral constrained equilibrium exists with a stationary structure. The equilibrium prices and allocation depend on the exogenous state of the economy and a measure of the wealth distribution.

### 3.1 The state space and definition

I define the normalized financial wealth of each agent by

\[
\omega_i^h = \frac{\sum_a (q_{a,t} + d_{a,t}) k_{a,t-1}^h + \sum_j \phi_{j,t}^h f_{j,t-1}}{\sum_a (q_{a,t} + d_{a,t}) K_a}.
\]

(23)

Let \(\omega(s_t) = (\omega^1(s_t), \ldots, \omega^H(s_t))\) denote the normalized financial wealth distribution. Then in equilibrium \(\omega(s_t)\) always lies in the \((H-1)\)-dimensional simplex \(\Omega\), i.e., \(\omega^h \geq 0\) and \(\sum_{h=1}^H \omega^h = 1\). \(\omega^h\)'s are positive because of the collateral constraint (7) that requires the value of each agent’s asset holdings to exceed the liabilities from their past financial assets holdings. And the sum of \(\omega^h\) equals 1 because of the asset market clearing and financial market clearing conditions. In Cao (2010), I show that, under conditions detailed in Subsection 3.2 below, there exists a Markov equilibrium over the compact state space \(S \times \Omega\), i.e., an equilibrium in which equilibrium prices and allocation depend only on the state \((s_t, \omega_t) \in S \times \Omega\).

\textsuperscript{26}Sandroni (2000) shows that if none of the agents has the correct beliefs, then only agents with beliefs closest to the truth survive, where distance is measured using entropy.
In particular, for each \((s_t, \omega_t) \in \mathcal{S} \times \Omega\), we need to find a vector of prices and allocation

\[
\nu_t \in \widehat{\mathcal{V}} = \mathbf{R}_+^H \times \mathbf{R}_+^{AH} \times \mathbf{R}_+^{JH} \times \mathbf{R}_+^A \times \mathbf{R}_+^J
\]

that consists of the consumers’ decisions: consumption of each consumer \((c^h (\sigma))_{h \in \mathcal{H}} \in \mathbf{R}_+^H\), real and financial asset holdings \((k^h_a (\sigma), \phi^j_h (\sigma))_{h \in \mathcal{H}} \in \mathbf{R}_+^{AH} \times \mathbf{R}_+^{JH}\); the prices of physical assets \((q_a (\sigma))_{a \in \mathcal{A}} \in \mathbf{R}_+^A\) and the prices of the financial assets \((p_j (\sigma))_{j \in \mathcal{J}} \in \mathbf{R}_+^J\). \(\nu\) must satisfy the market clearing conditions and the budget constraint of consumers bind. Moreover, for each future state \(s_{t+1}^+ \in \mathcal{S}\) succeeding \(s_t\), we need to find a corresponding wealth distribution \(\omega_{t+1}^+ \in \mathcal{V}\) such that for each household \(h \in \mathcal{H}\) the following conditions holds

a) For each \(s^+ \in \mathcal{S}\) succeeding \(s^27\)

\[
\omega^h_{s^+} = \frac{k^h \cdot (q^+_s + d^+_s) + \sum_{j \in \mathcal{J}} \phi^j_h \min \{b_j (s), k^j \cdot (q^+_s + d^+_s)\}}{\sum_a (q^+_s + d^+_s) \cdot K}.
\]  

b) There exist multipliers \(\mu^h_a \in \mathbf{R}_+^+\) corresponding to collateral constraints such that

\[
0 = \mu^h_a - q_a U_h' (c^h) + \beta_h E_h^h \{ (q^+_a + d^+_a) U_h' (c^h+) \}
\]

\[
0 = \mu^h_a \left( k^h_a + \sum_{j \in \mathcal{J} : \omega^j_h < 0} k^j_a \phi^j_h \right)
\]

\[
0 \leq k^h_a + \sum_{j \in \mathcal{J} : \omega^j_h < 0} k^j_a \phi^j_h.
\]

c) Define \(\phi^h_j (-) = \max (0, -\phi^h_j)\) and \(\phi^h_j (+) = \max (0, \phi^h_j)\), there exist multipliers \(\eta^h_j (+)\) and \(\eta^h_j (-) \in \mathbf{R}_+^+\) such that

\[
0 = \sum_{a \in \mathcal{A}} \mu^h_a k^j_a - p_j U_h' (c^h) + \beta^h E_h^h \{ f^+_j U_h' (c^h+) \} - \eta^h_j (-)
\]

\[
0 = -p_j U_h' (c^h) + \beta^h E_h^h \{ f^+_j U_h' (c^h+) \} + \eta^h_j (+)
\]

\[
0 = \phi^h_j (+) \eta^h_j (+)
\]

\[
0 = \phi^h_j (-) \eta^h_j (-).
\]

Condition a) guarantees that the future normalized wealth distributions are consistent with the current equilibrium decision of the consumers. Conditions b) and c) are the first-order conditions from the maximization problem (5) of the consumers. Given that the maximization problem is convex, the first-order conditions are sufficient for a maximum. As a result, as in Duffie, Geanakoplos, Mas-Colell, and McLennan (1994), a Markov equilibrium is a collateral constrained equilibrium.

Before continue, let me briefly discuss asset prices in a Markov equilibrium. We can rewrite that first-order condition with respect to asset holding (26) as

\[
q_a U_h' (c^h) = \mu^h_a + \beta_h E_h^h \{ (q^+_a + d^+_a) U_h' (c^h+) \} \geq \beta_h E_h^h \{ (q^+_a + d^+_a) U_h' (c^h+) \}.
\]

\(^{27}\)We use the vector product notation \(a \cdot b = \sum_i a_i b_i\).
By re-iterating this inequality we obtain

$$q_{a,t} \geq E_t^h \left\{ \sum_{r=1}^{\infty} \beta_n^r d_{t+r} \frac{U'_h(c^h_{t+r})}{U'_h(c^h_t)} \right\}.$$  

We have a strict inequality if there is a strict inequality $\mu^h_{a,t+r} > 0$ in the future. So the asset price is higher than the discounted value of the stream of its dividend because in future it can be sold to other agents, as in Harrison and Kreps (1978) or it can be used as collateral to borrow as in Fostel and Geanakoplos (2008). Proposition 2 shows some conditions under which collateral constraints will eventually be binding for every agent when they strictly differ in their belief. As a results, the prices of the real assets are strictly higher than the discounted value of their dividends.\(^{28}\)

Equation (26) also shows that asset $a$ will have collateral value when some $\mu^h_a > 0$, in addition to the asset’s traditional pay-off value weighted at the appropriate discount factors. Unlike in Alvarez and Jermann (2000), attempts to find a pricing kernel which prices assets using their pay-off value might prove fruitless because assets with the same payoffs but different collateral values will have different prices. This point is also emphasized in Geanakoplos’s papers.

Equation (27) implies that

$$p_j = \sum_{a \in \mathcal{A}} \mu^h_a k^j_a \frac{U'_h(c^h)}{U'_h(c^h)} + \beta_h E^h \left[ f^+_j \frac{U'_h(c^h)}{U'_h(c^h)} \right].$$

As in Garleanu and Pedersen (2010), the price of financial asset $j$ does not only depend on its promised payoffs in future states $\beta_h E^h \left[ f^+_j \frac{U'_h(c^h)}{U'_h(c^h)} \right]$ but also on its collateral requirements $\frac{\sum_{a \in \mathcal{A}} \mu^h_a k^j_a}{U'_h(c^h)}$.

### 3.2 Existence and Properties of Markov equilibrium

The existence proof is similar to the ones in Kubler and Schmedders (2003) and Magill and Quinzii (1994). We approximate the Markov equilibrium by a sequence of equilibria in finite horizon. There are three steps in the proof. First, using Kakutani fixed point theorem to prove the existence proof of the truncated T-period economy. Second, show that all endogenous variables are bounded. And lastly, show that the limit as T goes to infinity is the equilibrium of the infinite horizon economy.

To prove that the Markov equilibrium exists, we need to first show that there exists a compact set in which finite horizon equilibria lie. We need the following assumption

\(^{28}\)We can also derive a formula for the equity premium that depends on the multipliers $\mu$ similar to the equity premium formula in Mendoza (2010)
Assumption 6 There exist $\overline{c}, \underline{c} > 0$ such that

$$U_h(\overline{c}) + \max \left\{ \frac{\beta_h}{1 - \beta_h} U_h(\overline{c}), 0 \right\} \leq \min \left\{ \frac{1}{1 - \beta_h} \min_{s \in S} U_h(\underline{c}), \min_{s \in S} U_h(\underline{c}) \right\} \quad \forall h \in H. \quad (28)$$

and

$$U_h(\overline{c}) + \min \left\{ \frac{\beta_h}{1 - \beta_h} U_h(\overline{c}), 0 \right\} \geq \max \left\{ \frac{1}{1 - \beta_h} U_h(\overline{c}), U_h(\overline{c}) \right\} \quad \forall h \in H. \quad (29)$$

The intuition for (28) is detailed in the proof of Theorem 1; it ensures a lower bound for consumption. (29) ensures that prices of real assets are bounded from above. Both inequalities are obviously satisfied by log utility.

Lemma 1 Suppose Assumption 6 is satisfied then there is a compact set that contains the equilibrium endogenous variables constructed in Cao (2010) for every $T$ and every initial condition lying inside the set.

Proof. Appendix. $\blacksquare$

Theorem 2 Under the same conditions, a Markov equilibrium exists.

Proof. In Cao (2010), I show the existence of Markov equilibria for a general model also with capital accumulation. As in Kubler and Schmedders (2003), we extract a limit from the T-finite horizon equilibria. Lemma 1 guarantees that equilibrium prices and quantities are bounded as $T$ goes to infinity. The proof use an alternative definition of attainable sets and also corrects several errors in the Appendix of Kubler and Schmedders (2003). $\blacksquare$

As argued in the last subsection, any Markov equilibrium is a collateral constrained equilibrium. So Markov equilibria inherits all properties of collateral constrained equilibria. In particular, in a Markov equilibrium, every consumer survives, Theorem 1, and Markov equilibrium allocations are Pareto-inefficient if agents strictly differ in their beliefs, Corollary 18. Regarding asset prices, the construction of Markov equilibria provides us the following Proposition

Proposition 6 In contrast to the complete markets benchmark, in these Markov equilibria, asset prices can be history-dependent in the long run.

Proof. Proposition 4 shows that under complete markets, asset prices do depend on the wealth distribution but the wealth distribution converges in the long run, so asset prices only depend on the current exogenous state $s_t$. However, in a Markov equilibrium, the normalized

---

29This assumption is slightly different from the one in Kubler and Schmedders (2003) as $U_h$ might be negative, for example with log utility.
financial wealth distribution constructed in (23) constantly moves over time even in the long run. For example, if some agent $h$ with incorrect belief lose all her real asset holding due to leverage. Next period, she can always use her endowment to speculate in the real assets again. In this case, $\hat{\omega}_t^h$ will jump from 0 to a strictly positive number. So asset prices depend on the past realizations of the exogenous shocks, which determine the evolution of the normalized wealth distribution $\hat{\omega}_t$. ■

**Proposition 7** When the aggregate endowment is constant across states $s \in S$, and shocks are I.I.D., long run asset price volatility is higher in Markov equilibria than it is in complete markets equilibria.

**Proof.** Corollary 3 shows that, in the long run, in complete markets and the assumptions above, the economy converges to the one with homogenous beliefs because agents with incorrect beliefs will eventually be driven out of the markets and asset price $q_a(s^t)$ converges to prices independent of time and state. Hence, under complete markets, asset price volatility converges to zero in the long run. In Markov equilibrium, asset price volatility remains well above zero as the exogenous shocks constantly change the wealth distribution, which, in turn, changes asset prices. ■

There are two components of asset price volatility. The first and standard one comes from the volatility in the dividend process and the aggregate endowment. The second one comes from wealth distribution, when agents strictly differ in their beliefs. In general, it depends on the correlation of the two components, that we might have asset price volatility higher or lower under collateral constrained versus under complete markets. However, the second component disappears under complete markets because only agents with correct beliefs survive in the long run. Whereas, under collateral constraints, this component persists. As a result, when we shut down the first component, asset price is more volatile under collateral constraints than it is under complete markets in the long run. In general, the same comparison holds or not depending on the long-run correlation between the first and the second volatility components under collateral constrained markets.

## 4 Numerical Method

The construction of Markov equilibria in the last section also suggests an algorithm to compute them. The following algorithm is based on Kubler and Schmedders (2003). There is one important difference between the algorithm here and the original algorithm. The future wealth distributions are included in the current mapping $\rho$ instead of solving for them using sub-fixed-point loops. This innovation reduces significantly the computing time, given that solving for a fixed-point is time consuming in MATLAB. In section 5, as we seek to find the set of actively traded financial assets and the equilibrium leverage level in the economy, we need to know the future prices of the physical asset. As we know future wealth distributions, these future prices can be computed easily.

As in the existence proof, we look for the following correspondence

$$
\rho : S \times \Omega \longrightarrow \hat{\mathcal{V}} \times \hat{\Omega}^S \times \mathcal{L} \\
(s, \omega) \longmapsto (\hat{v}, \omega^+_s, \mu, \eta)
$$

(30)
\( \hat{v} \in \hat{V} \) is the set of endogenous variables excluding the wealth distribution as defined in (24). \((\omega^+_s)_{s \in S}\) are the wealth distributions in the \(S\) future states and \((\mu, \eta) \in \mathcal{L}\) are Lagrange multipliers as defined in subsection 3.1.

From a given continuous initial mapping \( \rho^0 = (\rho^0_1, \rho^0_2, \ldots, \rho^0_S) \), we construct the sequence of mappings \( \{\rho^n = (\rho^n_1, \rho^n_2, \ldots, \rho^n_S)\}_{n=0}^{\infty} \) by induction. Suppose we have obtained \( \rho^n \), for each state variable \((s, \omega)\), we look for

\[
\rho^{n+1}_s(\omega) = (\hat{v}_{n+1}, \omega^+_s, \mu_{n+1}, \eta_{n+1})
\]

that solves the first-order conditions (26), (27), market clearing conditions, and the consistency of the future wealth distribution (25).

We construct the sequence \( \{\rho^n\}_{n=0}^{\infty} \) on a finite discretization of \( S \times \Omega \). So from \( \rho^n \) to \( \rho^{n+1} \), we will have to extrapolate the values of \( \rho^n \) to outside the grid using extrapolation methods in MATLAB. Fixing a precision \( \delta \), the algorithm stops when \( \|\rho^{n+1} - \rho^n\| < \delta \).

There are two important details in implementing this algorithm: First, in order to calculate the \((n+1)\)-th mapping \( \rho^{n+1} \) from the \(n\)-th mapping, we need to only keep track of the consumption decisions \( c^h \) and asset prices \( q_h \) and \( p_j \). Even though other asset holding decisions and Lagrange multipliers might not be differentiable functions of the normalized financial wealth distribution, the consumption decisions and asset prices normally are.\(^{30}\) Relatedly, when there are redundant assets, there might be multiple asset holdings that implement the same consumption policies and asset prices.\(^{31}\) Second, if we choose the initial mapping \( \rho^0 \) as an equilibrium of the 1-period economy as in Subsection 3.2, then \( \rho^n \) corresponds to an equilibrium of the \((n+1)\)-period economy. I follow this choice in computing an equilibrium of the two agent economy presented below.

The algorithm to compute complete markets equilibria and is presented in Appendix A.

5 Asset price volatility and leverage

This section uses the algorithm just described to compute collateral constrained and complete markets equilibria and study asset price and leverage. To make the analysis as well as the numerical procedure simple, I allow for only one asset and two types of agents: optimists and pessimists. The general framework in Section 2 allows for a wide range of financial assets with different promises and collateral requirements. However, given that the total quantity of collateral is exogenously bounded, in equilibrium, only certain financial assets are actively traded. I choose a specific setting based on Geanakoplos (2009), in which I can find exactly which assets are traded. The setting requires that promises are state-incontingent and in each exogenous state there are only two possible future exogenous states. The only financial assets that are traded are the assets that allow maximum borrowing while keeping the payoff to lenders riskless. As noticed in Remark 3, the leverage level corresponding to this unique financial asset can be called the leverage level in the economy.

Endogenous financial assets interestingly generate the most volatility in the financial wealth distribution as agents borrow to the maximum and lose most of their financial wealth

\(^{30}\) See Brumm and Grill (2010) for an algorithm with adapted grid points that deals directly with non-differentiabilities in the policy functions.

\(^{31}\) See Cao, Chen, and Scott (2011) for such an example.
as they lose their bets but their wealth increases largely when they win. This volatility in the wealth distribution in turn feeds into asset price volatility.

To answer questions related to collateral requirements asked in the introduction, in Subsection 5.2.5, I allow regulators to control the sets of financial assets that can be traded. Given the restricted set, the endogenous active assets can still be determined. One special case is the extreme regulation that shuts down financial markets. There are surprising consequences of these regulations on the welfare of agents, on the equilibrium wealth distribution and on asset prices.

An endogenous set of traded assets also implies endogenous leverage which has been the object of interest during the current financial crisis. In order to match the observed pattern of leverage, i.e., high in good states and low in bad states, I introduce the possibility for changing volatility from one aggregate state to others. This feature is introduced in Subsection 5.2.6.

5.1 One asset economy

Consider a special case of the general model presented in Section 2. There are two exogenous states \( S = \{G, B\} \) and one single asset of which the dividend depends on the exogeneous state:

\[
d(G) > d(B).
\]

The state follows an I.I.D. process, with the probability of high dividends, \( \pi \), unknown to agents in this economy. The supply of the asset is exogenous and normalized to 1. Let \( q(s^t) \) denote the ex-dividend price of the asset at each history \( s^t = (s_0, s_1, \ldots, s_t) \).

To study the standard debt contracts, I consider the set of \( J \) of financial assets which promise state-independent payoffs next period. I normalize these promises to \( b_j = 1 \). Asset \( j \) also requires \( k_j \) units of the real asset as collateral. The effective pay-off is therefore

\[
f_{j,t+1}(s^{t+1}) = \min\{1, k_j \left(q(s^{t+1}) + d(s^{t+1})\right)\}.
\]

Due to the finite supply of the real asset, in equilibrium only a subset of the financial assets in \( J \) are traded. Determining which asset are traded allows us to understand the evolution of leverage in the economy (Remark 3). This is also important for computing collateral constrained equilibria in Subsection 4. It turns out that in some special cases, we can determine exactly which financial assets are traded. For example, Fostel and Geanakoplos (2008) and Geanakoplos (2009) argue that if we allow for the set \( J \) to be dense enough then in equilibrium the only financial asset traded in equilibrium is the one with the minimum collateral level to avoid default. This statement also applies for my general set up under the condition that in a history node, there are only two future exogeneous states. Proposition 8 below makes it clear. The proposition uses the following definition

**Definition 3** Two collateral constrained equilibria are equivalent if they have the same allocation of consumption to the consumers and the same prices of real and financial assets. They might differ in the consumers’ portfolios of real and financial assets.
Proposition 8  Consider a collateral constrained equilibrium and suppose in a history $s^t$, there are only two possible future exogenous states $s_{t+1}$. Let

$$ u = \max_{s^{t+1} | s^t} \left( q \left( s^{t+1} \right) + d \left( s_{t+1} \right) \right) $$

$$ d = \min_{s^{t+1} | s^t} \left( q \left( s^{t+1} \right) + d \left( s_{t+1} \right) \right) $$

and

$$ k^* = \frac{1}{d}. $$

We can find another collateral constrained equilibrium equivalent to the initial one such that only the financial assets with the collateral levels

$$ \max_{j \in J, k_j \leq k^*} k_j \text{ and } \min_{j \in J, k_j \geq k^*} k_j $$

are actively traded.

Proof. Intuitively, this proposition is true because with only two future states, two assets, a financial asset and the real asset, can effectively replicate the pay-off all other financial assets. But we need to make sure that we do not have to use short-selling in any replication. The detail of the proof is in Appendix B.

Imagine that the set $J$ includes all collateral requirements $k_j \in \mathbb{R}^+, k_j > 0$. Proposition 8 says, for any collateral constrained equilibrium, we can find an equivalent collateral constrained equilibrium in which the only financial asset with the collateral requirement exactly equals to $k^* (s^t)$ are traded in equilibrium. Therefore in such an equilibrium the only actively traded financial asset is riskless to its buyers. Let $p(s^t)$ denote the price of this financial asset. The endogenous interest rate is therefore $r(s^t) = \frac{1}{p(s^t)} - 1$. Proposition 8 is also useful to study financial regulations which correspond to choosing the set $J$ in Subsection 5.2.5.

Coming back to consumers, there are only two types of agents in this economy, optimists, $O$, and pessimists, $P$, each in measure one of identical agents. They differ in their belief: suppose agent $h \in \{O, P\}$ estimates the probability of high dividends as $\pi^h_G = 1 - \pi^h_B$. We suppose $\pi^O_G > \pi^P_G$, i.e. optimists always think that good states are more likely than the pessimists think they are. Each agent maximizes the inter-temporal utility (5) given their belief of the evolution of the aggregate state, and is subject to the budget constraint:

$$ c_t + q_t \theta_t + p_t \phi_t \leq c_t + (q_t + d_t) \theta_{t-1} + f_t \phi_{t-1}, $$

no short-sale constraint

$$ \theta_t \geq 0, $$

and collateral constraint

$$ \theta_t + \phi_t k_t^* \geq 0, $$

\(^32\)To apply the existence theorem 2 I need $J$ to be finite. But we can think of $J$ as a fine enough grid.

\(^33\)The uniqueness of actively traded financial assets established in Geanakoplos (2009) and He and Xiong (2011) is in the "equivalent" sense in Proposition 8.
for each \( h \in \{O, P\} \). At time \( t \), each agent choose to buy \( \theta_t \) units of real asset at price \( q_t \) and \( \phi_t \) units of financial asset at price \( p_t \). Moreover, Proposition 8 allows us to focus on only one level of collateral requirement \( k^h_t \). As a result, the set of collateral constraints (7) can be replaced by the constraints (33) and (34).

Given prices \( q \) and \( p \), this program yields solution \( c^h (s^t) , \theta^h_t (s^t), \phi^h_t (s^t) \). In equilibrium prices \( \{q_t (s^t)\} \) and \( \{p_t (s^t)\} \) are such that asset and financial markets clear, i.e.,

\[
\begin{align*}
\theta^O_t + \theta^P_t &= 1 \\
\phi^O_t + \phi^P_t &= 0
\end{align*}
\]

for each history \( s^t \).

With only one physical asset and one financial asset, the general formula for normalized financial wealth (23) translates into

\[
\omega^h_t = \frac{(q_t + d_t) \theta^h_{t-1} + f_t \phi^h_{t-1}}{q_t + d_t}.
\]

Again, due to the collateral constraint, in equilibrium, \( \omega^h_t \) must always be positive and

\[
\omega^O_t + \omega^P_t = 1.
\]

The pay-off relevant state space

\[
\{ (\omega^O_t, s_t) : \omega^O_t \in [0, 1] \text{ and } s_t \in \{G, B\} \}
\]

is compact.\(^{34}\) Section 2 showed the existence of collateral constrained equilibria under the form of Markov equilibria in which prices and allocations depend solely on that state defined above. Section 4 provides an algorithm to compute such equilibria. As explained in Proposition 6, the equilibrium asset prices depend not only on the exogenous state but also on the normalized financial wealth.

### 5.2 Numerical Results

In this subsection, I will chose plausible parameters to quantify the magnitude of the properties presented in Sections 2 and 3.2. These parameters are chosen such that the size of the financial sector is about 5\% (\( c^O \)) of the US economy, and the size of the housing markets (the real asset) is about 20\% of the US annual GDP. In particular, let

\[
\begin{align*}
\beta &= 0.95 \\
d (G) &= 1 > d (B) = 0.2 \\
U (c) &= \log (c),
\end{align*}
\]

and the beliefs are \( \pi^O = 0.9 > \pi^P = 0.5 \). Given the high discount factor, the algorithm takes about 2 hours to converge. To study the issue of survival and its effect on asset prices, \( ^{34} \)Given that the optimists prefer holding the physical asset, i.e., \( \theta^O_t > 0, \omega^O_t \) corresponds to the fraction of the asset owned by the optimists.
I assume that the pessimists have the correct belief, i.e., \( \pi = \pi^P = 0.5 \). Thus the optimists are over-optimistics.

I fix the endowments of the pessimists and the optimists at

\[
\begin{align*}
e^P &= \begin{bmatrix} 100 & 100.8 \end{bmatrix} \\
e^O &= \begin{bmatrix} 5 & 5 \end{bmatrix}.
\end{align*}
\]

The aggregate endowment is kept constant by choosing the pessimists' endowment to be state dependent.

### 5.2.1 Asset Prices

The numerical algorithms presented in Section 4 for collateral constrained and Appendix A for complete markets allow us to examine and compare the properties of asset prices under the two market arrangements. The numerical example here illustrates Propositions 4 and 6 on asset prices.

**Collateral Constrained Equilibrium:** I rewrite the budget constraint of the optimists (32) using normalized financial wealth, \( \omega_t^O \),

\[
c_t + q_t \theta^O_t + p_t \phi^O_t \leq e_t^O + (q_t + d_t) \omega^O_t.
\]

Therefore, their total wealth \( e_t^O + (q_t + d_t) \omega^O_t \) affects their demand for the asset. If non-financial endowment \( e_t^O \) of the optimists is small relative to price of the asset, their demand for the asset is more elastic with respect to their financial wealth \( (q_t + d_t) \omega^O_t \). In Cao (2010), I investigate this relationship by varying the endowment of the optimists.

Figure 1 plots the price of the physical asset as a function of the optimists' normalized financial wealth \( \omega^O_t \) for the good state \( s_t = G \) on the left axe, the solid line. The function for the bad state \( s_t = B \) is similar. Interest rate \( r \) is also endogenously determined in this economy, however most of the time it hovers around the common discount factors of the two agents, i.e. \( r(s^t) \approx \frac{1}{2} - 1 \).

The mechanism for the high slopes at the right and at the left are not the same. On the left side of the figure, when \( \omega^O_t \) is close to zero, the optimists are highly leveraged to buy the physical asset. If a bad shocks hits in the next period, they have to sell off their asset holdings to pay off their debts. Their next period financial wealth plummets and contributes to the fall in asset price. Consequently, we can see in Figure 1, the slope of the price function is steeper for \( \omega^O_t < 0.2 \). This also corresponds to the region in which the borrowing constraint binds, or is going to bind in the near future. Then when a bad shock hits the economy, that is \( s_{t+1} = B \), the optimists are forced to liquidate their physical asset holdings. This fire sale of the physical asset reduces its price and tightens the constraints further, thus setting off the vicious circle of falling asset price. Notice that, this channel also explains a flatter slope at the lowest level of financial wealth of the optimists as they hold less of the physical asset, the asset firesale has less bite. This dynamics of asset price under borrowing constraint corresponds to the "debt-deflation" channel in a small-open economy in Mendoza (2010). This example shows that the channel still operates when we are in a closed-economy with endogenous interest rate, \( r(s^t) \) as opposed to exogenous interest rates in Mendoza (2010) or in Kocherlakota (2000).
Figure 1: Asset Price and Asset Price Volatility Under Collateral Constraints

On the right side of the figure, the "debt-deflation" channel is not present as the collateral constraint is not binding or going to be binding in the near future. High asset price elasticity with respect to the normalized financial wealth of the optimists is due to their high exposure to the asset. As the optimists own most of the real asset, shocks to the dividend of the asset directly impacts the wealth, or cash in hand of the optimists. These shocks feed into large changes in the optimists’ marginal utilities, thus large changes in the price of the real asset because the optimists are the marginal buyers of the asset. Due to the same reason, long run asset price volatility is higher when the optimists cannot lever using the real asset, analyzed in Subsection 5.2.5.

Figure 1 also plots asset price volatility as function of the normalized financial wealth of the optimists, the dashed line on the right axe. This function shows that at low and medium levels of financial wealth of the optimists, asset price is inversely related to asset price volatility. This negative relationship between price level and price volatility has been observed in several empirical studies, for example, Heathcote and Perri (2011).

In order to study the dynamics of asset price, we need to combine the fact that asset price as a function of the normalized financial wealth shown in Figure 1 with the evolution of the exogenous state and the evolution of the "normalized financial wealth" distribution, $\omega_t^O$. Figure 2 shows the evolution of $\omega_t^O$. The left panel corresponds to the current good state $s_t = G$, and the right panel corresponds to the current bad state $s_t = B$. The solid lines represent next period normalized wealth of the optimists as a function of the current normalized wealth, if good shock realizes next period. The dashed lines represent the same function when the bad shock realize next period. I also plot the 45 degree lines for comparison. This figure shows that, in general, good shocks tend to increase and bad shocks tend to decrease the normalized wealth of the optimists. This is because the optimists bet more for the good state to happen (buy borrowing collateralized and investing into the real asset). A similar evolution of the wealth distribution holds for complete markets.

We can also think of normalized wealth as the fraction of the trees that the optimists
owns. When the current state is good, and the fraction is high, the optimists will get a lot of dividends from their tree holdings, due to consumption smoothing they will not consume all the dividends, but will use some part of the dividends to buy new trees, so we see that on the left panel, the tree holding of the optimists normally increases at high $\omega^O$. Similarly when the current state is bad, and the fraction is high, the optimists will sell off some of their tree holdings to smooth consumption. As a result, we see on the right panel that the tree holding of the optimists normally decreases at high $\omega^O$.

**Complete Markets Equilibrium:** In a complete markets equilibrium, as shown in Appendix A, the state variable is the consumption of the optimists. However, there is a one-to-one mapping from this state variable to a more meaningful state variable which is the relative wealth of the optimists, $\widehat{\omega}^O$, defined in (22). Similar to the collateral constrained equilibrium, this variable determines asset price and constantly changes as aggregate shocks hit the economy. Due to log utility and constant aggregate endowment, apply the general formula (21), we obtain the relationship between asset price and relative wealth

$$q(\widehat{\omega}) = \sum_{h \in \{O,P\}} \widehat{\omega}_h \frac{\beta}{1 - \beta} \left( \pi^h d(G) + (1 - \pi^h) d(B) \right).$$

(35)

This expression is the counterpart of Figure 1 for complete markets.\(^{35}\) Notice that at two extreme $\widehat{\omega}_t^O = 0$ or 1, we go back to the representative agent economy in which there are either only the optimists or the pessimists.\(^{36}\)

\(^{35}\)Unlike under collateral constraints, under complete markets the asset price function does not depend on the exogenous state $s_t$ due to the I.I.D. assumption and constant aggregate endowment.

\(^{36}\)When $\widehat{\omega}_t^O = 0$, asset price is the discounted value of average dividends evaluated at the pessimists’ belief

$$q^P = \frac{\beta}{1 - \beta} \left( \pi^P d(G) + (1 - \pi^P) d(B) \right),$$

$$q^P = \frac{\beta}{1 - \beta} \left( \pi^P d(G) + (1 - \pi^P) d(B) \right),$$

31
In this special case where the aggregate endowments are constant across states and shocks are I.I.D., applying Corollary 3, we have in the long run, with probability 1, the price of the real asset, \( q(s^t) \) converges to a constant

\[
\overline{q} = \frac{\beta}{1 - \beta} \left( P(G) d(G) + P(B) d(B) \right),
\]

i.e., asset price volatility decreases to zero in the long run. Another way to see this convergence, is to notice that the wealth distribution \((\tilde{b} = 0, \tilde{b} = 1)\) converges to \((0, 1)\); thus according to (35), \( q_t \) converges to \( q(0) \) given by (36).

In the short-run, however, the wealth distribution constantly changes as shocks hit the economy. Figure 3 depicts the evolution of the relative wealth distribution that determines the evolution of asset price under complete markets. This figure is the counterpart of Figure 2 under complete markets. Given that the aggregate endowment is constant, the transition of the wealth distribution does not depend on current aggregate state, unlike under collateral constraints. The optimists buy more Arrow-Debreu assets that deliver in the good future states and buy less Arrow-Debreu assets that deliver in bad future states. Therefore, when a good shock hits, the relative wealth of the optimists increases (solid line) and vice versa when a bad shock hits (dashed line). Notice that, as opposed to the Figure 2, \( \tilde{b}_O = 0 \) and \( \tilde{b}_O = 1 \) are two absorbing states. So the optimists disappear under complete markets but not under collateral constraints.

which is smaller than when \( \tilde{b}^O_t = 1 \), where asset price is the discounted value of average dividends evaluated at the optimists’ belief

\[
q^O = \frac{\beta}{1 - \beta} \left( \pi^O d(G) + (1 - \pi^O) d(B) \right) > q^P.
\]
This subsection illustrates the survival and disappearance results in Theorem 1 and Proposition 1. Figure 4 shows a realization of the financial wealth of the optimist starting at \( \omega^O = 0.1 \). The optimists always lever up to buy the real asset and often they will lose all their asset holdings (selling off their asset holdings to pay off their debt), in which case, their financial wealth reverts to zero. However, they can always use their non-financial endowment to come back to the financial markets by investing in the physical asset again (leveraged). Sometime they are lucky, that is, when the asset pays high dividends and its price appreciates, their financial wealth can increase rapidly. Given this dynamics, there exists a non-degenerate stationary financial wealth distribution of the optimists, shown in Figure 5. The spikes of the distribution (including the one at 0) shows that the financial wealth of the optimists often reverts to 0 and after that the optimists come back to the financial markets with certain levels of financial wealth, see Figure 2. This is in contrast with the results from the same simulation exercise for complete markets, where with probability 1 the wealth of the optimists will go to zero in the long run, thus the stationary distribution of the wealth of the optimists will be a degenerate mass at 0.

### 5.2.3 Asset Price Volatility

In this subsection, I compare asset price volatility under collateral constraints against the complete markets benchmark. I measure price volatility as one-period ahead standard deviation of price. This measure is the discrete time equivalence of the continuous instant volatility, see for example Xiong and Yan (2009). Figure 6 shows the evolution of asset price volatility under the two cases, the optimists with high or low non financial wealth. The figure shows that, in the short run, asset price is more volatile under complete markets than under collateral constraints. However, in the long run, as the optimists are driven out in the complete markets equilibrium; that makes asset price volatility converge to zero. This property does not hold in collateral constrained equilibria, the overly optimistic agents constantly
speculate on asset price using the same asset as collateral. Asset price becomes more volatile than in the complete markets equilibrium, given that the wealth of the optimists constantly change as they win or loose their bets. This result is an illustration of Proposition 7 because this economy has constant aggregate endowment.\footnote{Strikingly, the smaller the non-financial wealth of the optimist is, the higher the short-run asset price volatility in the collateral constrained equilibrium but the lower the short-run asset price volatility in the complete markets equilibrium. This is because, under complete markets, it takes less time to drive out the optimists if they have lower non-financial wealth. As we increase the non-financial wealth of the optimists, we increase the short-run volatility of asset price with complete markets and decrease the short-run volatility of asset price under collateral constraints. Above some certain level of non-financial wealth of the optimists, in the short-run asset price is more volatile under complete markets. But in the long run, the reverse inequality holds, see Cao (2010).}
5.2.4 The financial crisis 2007-2008

Geanakoplos (2009) argues that the introduction of credit default swap (CDS) triggered the financial crisis 2007-2008. The reason is that the introduction of CDS moves the markets close to being complete. CDS allow pessimists to leverage their pessimism about the assets. I do the same exercise here by simulating a collateral constrained equilibrium in its stationary state from time $t = 0$ until time $t = 49$. At $t = 50$ markets suddenly become complete. In Figure 7, the left panel plots asset price level and the right panel plots asset price volatility over time. The simulation shows that asset price decreases but asset price volatility increases in the short run after the introduction of CDS. The reason for the fall in asset price is that the pessimists can now "leverage their view" with a complete set of financial assets. The reason for increasing asset price volatility in the short run is the movement in the wealth distribution toward the long-run wealth distribution, which concentrates on pessimists. Nevertheless in the long run asset price volatility goes to zero when wealth distribution fully concentrates on the pessimists.

5.2.5 Regulating Leverage

Proposition 7 suggests that the variations in the wealth distribution drives up asset price volatility relative to the long run complete markets benchmark. It is then tempting to conclude that by restricting leverage, we can reduce the variation of wealth of the optimists, therefore reduce asset price volatility. However, this simple intuition is not always true. The reason is that, similar to collateral constraints, financial regulation may act as another device to protect the agents with incorrect beliefs from making wrong bets and disappear from the economy. The higher wealth increases their impact on asset prices, thus make asset prices more volatile.

To show this result, I consider an extreme form of financial regulation, that is, strictly
forbidding leverage. This corresponds to setting the set of financial assets $\mathcal{J}$ empty, or equivalently require infinite collateral $k \geq k_r$ where $k_r$ is very high. Figure 8 plots the volatility of asset price as functions of the financial wealth of the optimists in two cases, with financial markets and without financial markets\textsuperscript{38}. We can see that at the low levels of financial wealth of the optimists, asset price volatility is higher with financial markets than without financial markets. This is due to the debt-deflation, analyzed in Subsection 5.2.1, present under unregulated financial markets, but absent without financial markets. However, the opposite holds at higher level of financial wealth of the optimists. When the optimists hold a large fraction of the real asset, a drop in its dividends leads to a large drop in consumption and a large increase in the marginal utility of the optimists. As they are the marginal buyers, the increase in their marginal utility decreases the price of the real asset. This mechanism leads to high asset price volatility at the right side of Figure 8. But financial markets, allow the optimists to borrow to reduce the drop in consumption, thus mitigate the drop in the price of the real asset. It then makes asset price less volatile than when the financial markets are complete shut down. The numerical solution also shows that, without financial markets, the optimists always accumulate assets to move up to the high financial wealth (asset holding) region\textsuperscript{39}. This dynamics makes asset price more volatile without financial markets then it is with financial markets.

Figure 9 shows the Monte-Carlo simulation for an economy starting in good state and $\omega^O = 0$. The figure plots the evolution of the average of the normalized financial wealth of the optimists, left panel, and asset price volatility, right panel, over time (the solid lines represent the unregulated economy and the dashed lines represent the regulated economy). As discussed above, the wealth of the optimists remains low on average in the unregulated economy but increases to a permanently high level under regulation. Thus, initially asset

\textsuperscript{38}Without financial market, "financial wealth" is asset holding itself.

\textsuperscript{39}With leverage, the optimists will want to hold more of the real assets using leverage, but given that they have incorrect beliefs, they will tend to lose all their shares, and remain financially poor.
price volatility is higher in the unregulated economy than in the regulated economy. The reverse inequality holds, however, as over time, the wealth of the optimists increase more in the regulated economy than in the unregulated economy.

I conclude this part with two additional remarks. First, intermediate regulations can be computed using Proposition 8. If the regulator requires collateral $k \geq k_r$, then the proposition shows that in equilibrium, only the leverage level $\max (k^*_t, k_r)$ prevails. The numerical solutions for intermediate regulations confirm the conclusion in the paragraphs above. Second, regulation not only fails to reduce asset price volatility, it also reduces welfare of both types of agents as it reduces trading possibilities.

5.2.6 Dynamic leverage cycles

Even though the example in Subsection 5.2.3 generates high asset price volatility, leverage is not consistent with what we observe in financial markets: high leverage in good times and low leverage in bad times, as documented in Geanakoplos (2009).

In order to generate the procyclicality of leverage, I use the insight from Geanakoplos (2009) regarding aggregate uncertainty: bad news must generate more uncertainty and more disagreement in order to reduce equilibrium leverage significantly. The economy also constantly moves between low uncertainty and high uncertainty regimes. To formalize this idea, I assume that in the good state, $s = G$, next period dividend has low variance. However, when a bad shocks hits the economy, $s = GB$ or $BB$, the variance of next period dividend increases. In this dynamic setting, the formulation translates to a dividend process that depends not only on current exogenous shock but also on the last period exogenous shock. Therefore we need to use three exogenous shocks, instead of the two exogenous shocks in the last subsections:

$$s \in \{G, GB, BB\}.$$ 

Figure 10, left panel, shows that the good state, the variance of next period dividend is low, $d = 1$ or 0.8. However in bad states, the variance of next period dividend is higher, $d = 1$
or 0.2. The right panel of the figure shows the evolution through time of the exogenous states using Markov chain representation. Even though we have three exogenous states in this set-up, each state has only two immediate successors. So we can still use Proposition 8 to show that in any history there is only one leverage level in the economy.

The uncertainty structure generates high leverage at the good states $G$ and low leverage in bad states $GB$ and $BB$. Figure 11 shows this pattern of leverage. The dashed line represents leverage level in good states $s = G$ as a function of the normalized wealth distribution. The two solid lines represent leverage level in bad states $s = GB$ or $BB$.

In addition to the fact that uncertainty affects leverage emphasized in Geanakoplos (2009), we also learn from Figure 11 that financial wealth distribution is another important determinant of leverage. For example, we learn from the figure that leverage decreases dramatically from good states to bad states. However, in contrast to the static version in Geanakoplos (2009), changes in the wealth distribution do not amplify the decline in leverage from good states to bad states as leverage is relatively insensitive to the wealth distribution in bad states.

Moreover, this version of dynamic leverage cycles generates a pattern of leverage build-up
in good times. Good shocks increase leverage as they increase the wealth of the optimists relative to the wealth of the pessimists and leverage is increasing the wealth of the optimists. Figure 12 shows the evolution of the wealth distribution and leverage over time. The economy starts at good state and $\omega^O = 0$. It experiences 9 consecutive good shocks from $t = 1$ to 9 and two bad shocks at $t = 10, 11$ then another 9 good shocks from $t = 12$ to 19. This figure shows that, in good states, both the wealth of the optimists and leverage increase. However their wealth and leverage plunge when bad shocks hit the economy.

Notice, however, that even though leverage decreases significantly from 80 to 25 when a bad shock hits the economy, that leverage level is still too high compared to what observed during the last financial crisis. Gorton and Metrick (2010) document that leverage on some class of assets declined to almost 1 on some classes of assets. The reason that leverage is always very high in this model is due to high discount factor and $p_j \approx \beta$ in formula (3). Of course, there are some other channels absent in this paper that might have caused the rapid decline in leverage.

6 Conclusion

In this paper I develop a dynamic general equilibrium model to examine the effects of belief heterogeneity on the survival of agents and on asset price volatility under different financial markets structures. I show that, when financial markets are collateral constrained (endogenously incomplete), agents with incorrect beliefs survive in the long run. The survival of these agents leads to higher asset price volatility. This result contrasts with the frictionless complete markets case, in which agents holding incorrect beliefs are eventually driven out and as a result, asset prices and investment exhibit lower volatility.

In addition, I show the existence of stationary Markov equilibria in this framework with collateral constrained financial markets and with general production and capital accumula-
tion technology. I also develop an algorithm for computing the equilibria. As a result, the framework can be readily used to investigate questions about the interaction between financial markets and the macroeconomy. For instance, it would be interesting in future work to apply these methods in calibration exercises using more rigorous quantitative asset pricing techniques, such as in Alvarez and Jermann (2001). This could be done by allowing for uncertainty in the growth rate of dividends rather than uncertainty in the levels, as modeled in this paper, in order to match the rate of return on stock markets and the growth rate of aggregate consumption. Such a model would provide a set of moment conditions that could be used to estimate relevant parameters using GMM as in Chien and Lustig (2009). A challenge in such work, however, is that finding the Markov equilibria is computationally demanding. I follow that path in Cao, Chen, and Scott (2011).

A second avenue for further research is to examine more normative questions in the framework developed in this paper. My results suggest, for example, that financial regulation aimed at reducing asset price volatility should be state-dependent, as conjectured by Geanakoplos (2009). It would also be interesting to consider the effects of other intervention policies, such as bail-out or monetary policies.
Appendices

Appendix A: The analysis of survival, disappearance, and asset pricing under complete markets.

Proof of Proposition 1. This is an application of Proposition 5 in Sandroni (2000). We need to show that, for any agent \( h \) with incorrect belief, there exists \( l \) and \( \epsilon > 0 \) such that for any path \( \{s^t\} \)

\[
d_t \left( P_{s_t}^h, P_{s_t} \right) > \epsilon. \tag{37}
\]

Indeed, given that \( \pi^h \neq \pi \), there exists an \( s^* \) and \( s^{**} \) such that \( \pi^h (s^*, s^{**}) \neq \pi (s^*, s^{**}) \). Now Assumption 1 implies that for each \( s \in S \) there exists \( n \) such that \( \pi (n) (s, s^*) > 0 \). Then (1) then implies that \( \pi (n) (s, s^*) > 0 \). Let \( l \) be the maximum of these \( n \)'s over \( S \). Given the finiteness of \( S \), we can find \( \epsilon \) that satisfies (37).

Proof of Proposition 4. Let \( I \) denote the set of agents with the correct belief. The asset price is the presented discounted value of dividends weighted by the stochastic discount factor:

\[
q_a (s^t) = \sum_{r=1}^{\infty} P_h (s^{t+r} | s^t) \beta^r \frac{U'_{h} (c_{h} (s^{t+r}))}{U'_{h'} (c_{h'} (s^{t+r}))} \] 

or

\[
\frac{U'_{h} (c_{h} (s^{t+r}))}{U'_{h'} (c_{h'} (s^{t+r}))} = \frac{P_{h'} (s^{t+r}) U'_{h'} (c_{h'} (s^{t+r}))}{P_{h} (s^{t+r}) U'_{h'} (c_{h'} (s^{t+r}))}. \tag{39}
\]

For \( h \) and \( h' \in I \), we have, \( P_{h'} (s^{t+r}) = P_{h} (s^{t+r}) \), so

\[
\frac{U'_{h} (c_{h} (s^{t+r}))}{U'_{h'} (c_{h'} (s^{t+r}))} = \frac{U'_{h} (c_{h} (s^0))}{U'_{h'} (c_{h'} (s^0))}. \tag{40}
\]

Let

\[
C_I (s^{t+r}) = \sum_{h \in I} c_{h} (s^{t+r}).
\]

Using the equations (39), we can solve for \( c_{h} \) as functions of \( C_I \).

\[
c_{h} (s^{t+r}) = C_{h} (C_I (s^{t+r})) \text{ for each } h \in H.
\]

Applying Proposition 1, we have, almost surely,

\[
\lim_{t \to \infty} |C_I (s^t) - e(s_t)| = 0
\]

or

\[
\lim_{t \to \infty} |c_{h} (s^t) - C_{h} (e(s_t))| = 0. \tag{40}
\]

\[\text{See Stokey and Lucas (1989) for standard notations with transition matrices.}\]
Let
\[
\overline{q}_a (s_t) = \sum_{r=1}^{\infty} P \left( s^{r+r} | s^t \right) \beta^r \frac{U'_h (C_h (e (s_t)))}{U'_h (C_h (e (s_t)))} d_a (s_{t+r}) \text{ for any } h \in \mathcal{H}.
\] (41)

Finally, (38), (41), and (40) implies
\[
\lim_{t \to \infty} \sup_{r \geq 0} \left| q_a (s^{t+r}) - \overline{q}_a (s_{t+r}) \right| = 0.
\]

This is asset price in an complete markets economy. \(41\)

**Closed form solution in Example 1.** The natural state variable is the wealth distribution
\[
w_h (s^t) = \sum_{r=0}^{\infty} \sum_{s^{r+r} | s^t} p \left( s^{r+r} | s^t \right) c_h (s^{r+r}).
\]

In case of log utility,
\[
c_h (s^t) = (1 - \beta) w_h (s^t)
\]

The good market clearing condition implies
\[
e (s_t) = \sum_{h \in \mathcal{H}} c_h (s^t) = (1 - \beta) \sum_{h \in \mathcal{H}} w_h (s^t)
\]

We have
\[
q_a (s^t) = \sum_{r=1}^{\infty} \sum_{s^{r+r} | s^t} p \left( s^{r+r} | s^t \right) d_a (s_t)
\]

where, also given log utility,
\[
p \left( s^{r+r} | s^t \right) = \beta^r P_h \left( s^{r+r} | s^t \right) \frac{w'_h (s^{r+r})}{w'_h (s^t)} = \beta^r P_h \left( s^{r+r} | s^t \right) \frac{c_h (s^t)}{c_h (s^{r+r})}
\]
\[
= \frac{\beta^r \sum_{h \in \mathcal{H}} P_h \left( s^{r+r} | s^t \right) c_h (s^t)}{\sum_{h \in \mathcal{H}} c_h (s^{r+r})} = \frac{\beta^r \sum_{h \in \mathcal{H}} P_h (s^{r+r} | s^t) c_h (s^t)}{e (s^{r+r})}
\]
\[
= \sum_{h \in \mathcal{H}} c_h (s^t) \beta^r P_h \left( s^{r+r} | s^t \right) \frac{1}{e (s^{r+r})} = \sum_{h \in \mathcal{H}} (1 - \beta) w_h (s^t) \beta^r P_h \left( s^{r+r} | s^t \right) \frac{1}{e (s^{r+r})}.
\]

So finally
\[
\frac{q_a (s^t)}{e (s_t)} = \sum_{h \in \mathcal{H}} \omega_h^t (s^t) \left\{ \sum_{r=1}^{\infty} \sum_{s^{r+r} | s^t} \beta^r P_h \left( s^{r+r} | s^t \right) \frac{d(s_{t+r})}{e (s_{t+r})} \right\},
\]

where
\[
\omega_h^t (s^t) = \frac{(1 - \beta) w_h (s^t)}{e (s_t)} = \frac{w_h (s^t)}{\sum_{h' \in \mathcal{H}} w_{h'} (s^t)}
\]

**Numerical procedure for complete markets.** The state space should be
\[
(s, \{c_h\}_{h \in \mathcal{H}})
\]

\(41\)See Ljungqvist and Sargent (2004) for a standard treatment of asset pricing under complete markets with similar notations but with homogeneous beliefs.
such that
\[ \sum_{h \in \mathcal{H}} c_h = e(s) \]

We find the mapping \( \rho \) from that state space into the set of current prices and consumption levels \( \{q_a\}_{a \in \mathcal{A}} \), future consumptions \( \{(e^+_h)_{h \in \mathcal{H}}\}_{s \in S} \), and \( \{p_s\}_{s \in S} \) the Arrow-Debreu state prices. There are therefore \( A + (S + 1)H + S \) unknowns.

For each \( a \in \mathcal{A} \) we have \( A \) equations
\[ q_a = \sum_{s} p_s (q^+_a + d^+_a) \]

Regarding \( p_s \), the inter-temporal Euler equation implies
\[ p_s = \beta \pi^h(s, s^+) \frac{U'_h(e^+_h)}{U'_h(c_h)} \]

that give \( SH \) equations and finally
\[ \sum_{h} e^+_h = \sum_{h} e^+_h + \sum_{a \in \mathcal{A}} e^+_a K_a \]
\[ = e^+ \]

which give other \( S \) equations. With these \( A + (S + 1)H + S \) equations, we can solve for the \( A + (S + 1)H + S \) unknowns. That solution determines the mapping \( T \).

In order to find an equilibrium corresponding to an initial asset holdings \((\theta_{h,a})_{h \in \mathcal{H}, a \in \mathcal{A}}\) we find the value of stream of consumption and endowment of each consumers
\[ w^h_c = c_h + \sum_{s \in S} p_s w^{h+}_c (s) \]
and
\[ w^h_e = e_h + \sum_{s \in S} p_s w^{h+}_e (s) \]

Then we solve for \( H \) unknowns \((c_h)_{h \in \mathcal{H}}\) using \( H \) equations
\[ w^h_c = w^h_e + \sum_{a \in \mathcal{A}} \theta_{h,a} q_a. \]

Appendix B: One asset economy

**Proof of Proposition 8.** Let \( k_d = \max_{j \in \mathcal{J}, k_j \leq k^*} k_j \). Let \( p_d \) denote the price of the financial assets with collateral requirement \( k_u \). Suppose that there is another financial asset in \( \mathcal{J} \) that is actively traded and have collateral requirement \( k \leq k^* \). Then by definition \( k < k_d \). Let \( p_k \) denote the price of the financial asset. We have two cases:

Case 1) \( \frac{1}{\bar{a}} \geq k_d > k > \frac{1}{\bar{a}} \): Consider the optimal portfolio choice of a seller of financial asset \( k \). The pay-off from selling the asset is \((ku - 1, 0)\) and she has to pay \( kq - p_k \) in cash: she buys \( k \) units of the real asset but she get \( p_k \) from selling the financial asset. So the return on the financial asset is \( \frac{ku - 1}{kq - p_k} \) (when good state happens). Similarly, the return on selling financial asset \( k_d \) is \( \frac{k_d u - 1}{k_d q - p_d} \). If there are sellers for financial asset \( k \), it then implies that
\[ \frac{ku - 1}{kq - p_k} > \frac{k_d u - 1}{k_d q - p_d}. \]
or equivalently
\[ p_k \geq \frac{ku - 1}{kd u - 1} p_d + \frac{k_d - k}{kd u - 1} q, \] (42)
otherwise, sellers will strictly prefer selling financial asset \( k_d \) to financial asset \( k \). Now from the perspective of the buyers of the financial assets, the pay-off of financial asset \( k \) is \((1, kd)\). We can write this pay-off as a portfolio of financial asset \( k_d \) and the real asset:
\[ \begin{bmatrix} 1 \\ kd \end{bmatrix} = \frac{ku - 1}{kd u - 1} \begin{bmatrix} 1 \\ k_d d \end{bmatrix} + \frac{k_d - k}{kd u - 1} \begin{bmatrix} u \\ d \end{bmatrix}. \] (43)
As a result, if there are buyers for financial asset \( k \), we must have
\[ p_k \leq \frac{ku - 1}{kd u - 1} p_d + \frac{k_d - k}{kd u - 1} q \] (44)
otherwise the buyers will buy the portfolio (43) of financial asset \( k_d \) and the real asset instead. Thus we have both (42) and (44) happen with equality if financial asset \( k \) is actively traded. Armed with the equality, we can now prove the proposition. Consider each pair of seller and buyer of a unit of financial asset \( k \): the buyer buy one unit and the seller sells one unit of financial asset \( k \) at the same time is required to buy \( ku \) units of the real asset. We alter the their portfolios in the following way: instead of buying one unit of financial asset \( k \), the buyer buys \( ku \) units of financial asset \( k_d \) from the seller and \( k_d \) of the real asset. Given (43) and the equality (44), this changes in portfolio let the consumption and future wealth of the buyer unchanged. Now the seller instead of selling one unit of financial asset \( k \), sells \( \frac{ku - 1}{kd u - 1} \) units of financial asset \( k_d \) and holds \( \frac{k_d - k}{kd u - 1} k_d \) units of the real asset as collateral. Similarly, due to the equality (42), this transaction costs the same and yields the same returns to the seller compared to selling one unit of financial asset \( k \). So the consumption and the future wealth of the seller remains unchanged. Now we just need to verify that the total quantity of the real asset used remain unchanged. Indeed this is true because
\[ \frac{k_d - k}{kd u - 1} + \frac{ku - 1}{kd u - 1} k_d = k. \]

Case 2) \( \frac{1}{u} \geq k \): Financial asset \( k \)'s pay-off to the buyers is \( k (u, d) \) and to the seller is 0. So the financial asset is essential the real asset. This implies, \( p_k = kq \). The proposition follows immediately.
Now let \( k_u = \min_{j \in J, k_j \geq k^*} k_j \). Let \( p_u \) denote the price of the financial asset \( p_u \). The proof of the proposition is similar. First, we show that the price of any actively traded financial asset \( k \) with \( k \geq k^* \) is \( p_k = p_u \) and we can alter the portfolio of the buyers and sellers of financial asset \( k \) to transfer all the trade in the asset to financial asset \( k_u \). □
References


