Financial Innovation, the Discovery of Risk, and the U.S. Credit Crisis

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Abstract

Uncertainty about the true riskiness of new financial products was an important factor behind the U.S. financial crisis. We show that a boom-bust cycle in credit, asset prices and consumption characterizes the equilibrium dynamics of a model in which agents learn “by observation” the true riskiness of a new financial environment. Early realizations of states with high ability to leverage a fixed asset (e.g. housing) into debt turn agents overly optimistic about the probability of persistence of a high leverage regime. Debt, leverage and asset prices rise more than they would under rational expectations based on the true probabilities of switching across high and low leverage regimes. Conversely, the first realization of the low leverage state turns agents unduly pessimistic about future credit prospects, causing sharp declines in credit, asset prices and consumption. The model can account for roughly 90 percent of the rise in net credit liabilities of U.S. households and 1/3 of the increase in housing prices between 1997-2006. In addition, it predicts collapses in consumption and asset prices in 2007.

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1 Introduction

A key factor behind the U.S. financial crisis was the large expansion of credit and leverage driven by the introduction of new financial instruments that “securitized” the payment streams generated by a wide variety of assets, particularly mortgages. This process started with the gradual introduction of collateralized debt obligations (CDOs) in the 1980s, but became significantly more important since the mid 1990s with the introduction of collateralized mortgage obligations (CMOs) and insurance contracts on the payments of CDOs and CMOs known as credit default swaps (CDSs). In addition, “synthetic securitization” allowed third parties to trade these securities as bets on the corresponding income streams, without being a party to the actual underlying loan contracts. By the end of 2007, the market of CDSs alone was worth about $45 trillion (or 3 times U.S. GDP). Between 1996 and 2006, the net credit liabilities of U.S. households and non-profit organizations grew from 35 to 70 percent of GDP.

Several factors were at play in causing the securitization boom that ended with the financial crash, including moral hazard incentives in financial markets, particularly mortgages, and the lack of government supervision and regulation of the new and more complex securities, such as CDSs. In this paper, however, we focus on a different factor, namely the “natural” underpricing of the risk associated with the new financial instruments. Undervaluing the risk was natural because of the lack of data on the default and performance records of the new financial instruments, and because the strategy of “layering of (mispriced) risk” justified the belief that the new instruments were so well diversified that they were virtually risk free. The latter was being attained by using portfolio models to combine top tranches of AAA-rated assets with tranches containing more risky assets—under the key assumption that the risk of all of these assets was priced correctly. As Drew (2008) described it: “The computer modelers gushed about the tranches. The layers spread out the risk. Only a catastrophic failure would bring the structure crashing down, and the models said that
We examine in this paper the macroeconomic implications of a process of financial innovation with these characteristics. In particular, we propose a model in which the true riskiness of a new financial environment can only be discovered with time, and financial innovation begins with a period of optimism that leads agents to underestimate the possibility of a deterioration in credit conditions. In this model, agents face a credit constraint that hampers their ability to leverage by limiting debt not to exceed a fraction of the market value of their holdings of a fixed asset (e.g. land or housing).

Financial innovation is modelled as a structural change that rises the leverage limit, thus moving the economy to a “high leverage” state. Agents know that two financial regimes can materialize at any point in time: one in which high leveraging continues, and one in which there is a switch back to the pre-financial-innovation leverage limit (i.e. “low leverage”). They do not know the true riskiness of the new financial environment, because they lack data with which to estimate accurately the true regime-switching probabilities across high and low leverage states. They are
Bayesian learners, however, and so they learn over time as they observe regime realizations, and in
the long run their beliefs converge to the true regime-switching probabilities. Hence, in the long
run the model converges to the rational expectations (RE) solution, with the risk of the financial
environment priced correctly. In the short run, however, optimal plans and asset prices deviate
from the RE equilibrium, because beliefs differ from those of the RE solution and this leads to a
mispricing of risk.

Quantitative analysis shows that the process of discovery of risk has important effects on macroeco-
nomic aggregates, and can lead to a period of booming credit and land prices, followed by a sharp
and sudden collapse. The priors that agents start with when financial innovation occurs, and the
financial regime they observe in the periods that follow, matter significantly for macroeconomic
dynamics. In particular, if the priors are not overly pessimistic and the early regime realizations
are consistent with a high leverage regime, agents become optimistic. In this “optimistic phase,”
debt, leverage and collateral values (i.e. land prices) rise significantly above what the RE equi-
librium predicts. Conversely, when agents observe the first realization of the low leverage regime
after the optimistic phase, they respond with a sharp correction in their beliefs and become unduly
pessimistic, causing sharp downward adjustments in credit, land prices and consumption. Thus,
the process of discovery of true financial risk produces a credit boom followed by a bust.

In this setup, the degree of optimism generated in the optimistic phase is at its highest just before agents observe the first realization of the low leverage regime. This occurs because, when the new financial environment (which switches between high and low leverage states) is first introduced, agents cannot rule out the possibility of the high leverage regime being absorbent until they experience the first realization of the low leverage state.

We model learning following the approach proposed by Cogley and Sargent (2008). They offer an explanation of the equity premium puzzle by modelling a period of persistent pessimism caused by the Great Depression. They assume high and low states for consumption growth, with the true transition probabilities across these states unknown. Agents learn the true probabilities over time as they observe (without noise) the realizations of consumption growth. Similarly, in our setup, the true probabilities of switching across leverage regimes are unknown, and agents learn about them over time.

This paper is also related to the broader macro literature on the macroeconomic implications of learning. Most of this literature focuses on learning from noisy signals (see, for example, Blanchard et al. (2008), Boz (2009), Boz, Daude and Durdu (2008), Edge et al. (2007), Lorenzoni (2009), Nieuwerburgh and Veldkamp (2006)). The informational friction in these models typically stems from signal extraction problems requiring the decomposition of signals into a persistent component and a noise component. The informational friction in models like ours and Cogley and Sargent’s (2008) is fundamentally different because there is no signal extraction problem. Agents observe realizations of the relevant variables without noise. Instead, there is imperfect information about the true distribution of these variables. The financial innovations that led to the U.S. credit crisis provide a natural laboratory to study the effects of this class of learning models, because the new financial products clearly lacked the time-series data needed to infer the true probability of “catastrophic failure” of credit markets (i.e. the probability of switching to a low leverage regime).

The credit constraint used in our model is similar to those widely examined in the macro literature on financial frictions and in the international macro literature on Sudden Stops. When these constraints are used in RE stochastic environments, precautionary savings reduce significantly the long-run probability of states in which the constraints are binding (see Mendoza (2010) and Durdu, Mendoza and Terrones (2009)). In our learning model, however, agents have significantly weaker incentives for building precautionary savings than under rational expectations, until they attain the long run in which they have learned the true riskiness of the financial environment. Since
agents borrow too much during the optimistic phase, the economy is vulnerable to suffer a large credit crunch when the first switch to a regime with low leverage occurs.

Our credit constraint also features the “systemic credit externality” present in several models of financial crises. In particular, agents do not internalize the implications of their individual actions on credit conditions because of changes in equilibrium prices, and this leads to “overborrowing” relative to debt levels that would be acquired without the externality. Studies on overborrowing like those by Uribe (2006), Korinek (2008) and Bianchi (2009) explore whether credit externalities can generate excessive borrowing in decentralized equilibria relative to the social optimum. Uribe (2006) shows examples of environments in which overborrowing does not occur, and Korinek (2008) and Bianchi (2009) provide examples in which it does. Bianchi (2009) shows quantitatively that overborrowing can range from moderate to negligible depending on the elasticity of substitution of demand for the goods relevant for determining the price at which income used as collateral is valued, relative to the unit of denomination of debt contracts. Our paper makes two contributions to this line of research. First, we show that the discovery of risk generates sizable overborrowing (relative to the RE decentralized equilibrium), because of the unduly optimistic expectations of agents during the optimistic phase of the learning dynamics. This remains the case even in variants of our model with credit constraints that do not include the credit externality. Second, we provide the first analysis of the interaction between the credit externality and the underpricing of risk driven by a process of “risk discovery.” Thus, while we emphasize the relevance of the discovery of risk, the model does incorporate another underlying source of overborrowing associated with the credit market failure typical of models with credit externalities.

Our work is also related to the literature on credit booms. The stylized facts documented by Mendoza and Terrones (2008) show that credit booms have well-defined cyclical patterns, with the peak of credit booms preceded by periods of expansion in credit, asset prices, and economic activity followed by sharp contractions. Most of the models of financial crises, however, emphasize mechanisms that amplify downturns and explain crashes but leave booms unexplained. In this regard, our model aims to explain both the boom and the bust phase of credit cycles.

The remainder of this paper proceeds as follows: Section 2 describes the model and the learning process. Section 3 examines the model’s quantitative implications. Section 4 concludes.
2 A Model of Financial Innovation with Learning

We study a representative agent economy in which risk-averse individuals formulate optimal plans facing exogenous income volatility. The risk of income fluctuations cannot be fully diversified because asset markets are incomplete. Individuals have access to two assets: a non-state contingent bond and an in fixed aggregate supply (which we refer to interchangeably as land or housing). The credit market is imperfect because individuals’ ability to borrow is limited not to exceed a fraction \( \kappa \) of the market value of their land holdings. That is, \( \kappa \) imposes an upper bound on the agents’ leverage ratio.

The main feature that differentiates our model from other models with credit frictions is the assumption that agents have imperfect information about the regime-switching probabilities that drive fluctuations in \( \kappa \).\(^1\) We consider a situation in which financial innovation starts with an initial shift from a low leverage regime (\( \kappa^l \)) to a regime with higher ability to leverage (\( \kappa^h \)). Agents do not know the true regime-switching probabilities between \( \kappa^l \) and \( \kappa^h \) in this new financial environment. They are Bayesian learners, so in the long run they learn these true probabilities and can form rational expectations. In the short run, however, they form their expectations with the posteriors they construct as they observe actual realizations of \( \kappa \). Hence, they “discover” the true riskiness of the new financial environment only after they have observed a sample with enough regime realizations and regime switches to estimate the true regime-switching probabilities accurately.

We assume throughout that the risk-free interest rate is exogenous in order to keep the interaction between financial innovation and learning tractable. At the aggregate level, this assumption corresponds to an economy that is small and open with respect to world capital markets. This is in line with recent evidence suggesting that in the era of financial globalization even the U.S. risk-free rate has been significantly influenced by outside factors, such as the surge in reserves in emerging economies and the persistent collapse of investment rates in South East Asia after 1998 (see Warnock and Warnock (2006), Bernanke (2005), Durdu et al. (2009), Mendoza et al. (2009)). Moreover, post-war data also show that, while pre-1980s the United States was in virtual financial autarky, because the fraction of net credit of U.S. nonfinancial sectors financed by the rest of the world was close to zero, about 1/2 of the surge in net credit since the mid-1980s was financed by the rest of the world (see Mendoza and Quadrini (2009)). Alternatively, our setup can be viewed

\(^1\)In previous work we studied a similar informational friction but in a setup in which the credit constraint does not depend on market prices. In that scenario, the distortions produced by the learning process in the aftermath of financial innovation do not interact with the credit externality present in the model we study here.
as a partial equilibrium model of the U.S. economy that studies the effects of financial innovation taking the risk-free rate as given, as in Corbae and Quintin (2009).

2.1 Agents’ Optimization Problem

Agents acting atomistically in competitive markets choose consumption \((c_t)\), land holdings \((l_{t+1})\) and holdings of one-period discount bonds \((b_{t+1})\), taking as given the price of land \((q_t)\) and the gross real interest rate \((R)\) so as to maximize a standard intertemporal utility function with constant relative risk aversion (CRRA):

\[
E_0^s \left[ \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \right]
\]  

(1)

It is critical to note that \(E_t^s\) represents the expectations operator conditional on the representative agent’s beliefs formulated with the information available up to and including date \(t\). As we explain below, these beliefs will differ in general from the rational expectations formulated with perfect information about the persistence of the financial regime ruling in the economy, which are denoted \(E_t^a\).

The agents’ budget constraint is:

\[
c_t = (z_t g(l_t) + q_t l_t) (1 - \tau) - q_t l_{t+1} - \frac{b_{t+1}}{R} + b_t + \Upsilon_t
\]  

(2)

Agents operate a neoclassical production function \(g(l_t)\) subject to a stochastic TFP shock \(z_t\). The government levies taxes on income from production, \(z_t g(l_t)\), and capital gains, \(q_t l_t\), at a time-invariant rate \(\tau\) and rebates the revenue as lump sum transfers \(\Upsilon_t\) (i.e. the government budget constraint is simply \(\Upsilon_t = \tau (z_t g(l_t) + q_t l_t)\)). This tax is introduced so that the model can support an arbitrage condition for the returns on land and bonds consistent with existing estimates of the value of land as a share of GDP, the factor share of land in production, and the risk-free rate. As we show later in our calibration exercise, the resulting tax rates in the order of 15-25% seem realistic given existing estimates of effective capital income tax rates and statutory tax rates (see Mendoza, Razin and Tesar (1994).

TFP shocks follow an exogenous discrete Markov process (which can be enriched to include also interest rate shocks). For these shocks, we can assume that agents know their true joint Markov process without informational frictions, or alternatively we can assume that they are also affected by imperfect information. If agents know their true process, they know the Markov transition
matrix \( \pi(z_{t+1}|z_t) \) and the corresponding set \( Z \) of \( M \) possible realizations of \( z \) at any point in time (i.e. \( z_t \in Z = \{z_1, z_2, ..., z_M\} \)).

Credit markets are imperfect. In particular, agents must comply with the following collateral constraint that limits the value of debt (given by \( b_{t+1}/R \) since \( 1/R \) is the price of discount bonds):

\[
\frac{b_{t+1}}{R} \geq -\kappa_t q_t l_{t+1} \tag{3}
\]

In this constraint, \( \kappa_t \) is a random variable that follows a “true” Markov process characterized by a standard two-point regime-switching process with regimes \( \kappa^h \) and \( \kappa^l \), with \( \kappa^h > \kappa^l \), and transition probabilities given by \( F^a = p^a(\kappa_{t+1}|\kappa_t) \).\(^2\) The continuation transition probabilities are denoted \( F^a_{hh} \equiv p^a(\kappa_{t+1} = \kappa^h|\kappa_t = \kappa^h) \) and \( F^a_{lh} \equiv p^a(\kappa_{t+1} = \kappa^l|\kappa_t = \kappa^h) \), and the switching probabilities are \( F^a_{hl} = 1 - F^a_{hh} \) and \( F^a_{lh} = 1 - F^a_{ll} \). The long run probabilities of the high and low leverage regimes are \( \Pi^h = F^a_{lh}/(F^a_{lh} + F^a_{hh}) \) and \( \Pi^l = F^a_{hl}/(F^a_{hl} + F^a_{hh}) \) respectively, and the corresponding mean durations are \( 1/F^a_{hl} \) and \( 1/F^a_{lh} \). The long-run unconditional moments of \( \kappa \) are the following:

\[
E^a[\kappa] = \frac{(F^a_{hh}\kappa^h + F^a_{lh}\kappa^l)}{(F^a_{hh} + F^a_{lh})} \tag{4}
\]

\[
VAR(\kappa) = \Pi^h(\kappa^h)^2 + \Pi^l(\kappa^l)^2 - (E[\kappa])^2 \tag{5}
\]

\[
AUTOCORR(\kappa) = \frac{F^a_{ll} - F^a_{hl}}{F^a_{hh} - F^a_{lh}} \tag{6}
\]

As explained earlier, agents know \( \kappa^h \) and \( \kappa^l \) but do not know \( F^a \). Hence, they make decisions based on their individual beliefs \( F^s = p^s_t(\kappa_{t+1}|\kappa_t) \), which evolve over time.\(^3\) Using \( \mu \) to denote the Lagrange multiplier on the credit constraint, the agents’ optimality conditions for bonds and land are:

\[
u'(c_t) = \beta RE^s_t \left[ u'(c_{t+1}) \right] + \mu_t \tag{7}
\]

\[
q_t(u'(c_t) - \mu_t \kappa_t) = \beta (1 - \tau) E^s_t \left[ u'(c_{t+1}) (z_{t+1} g'(l_{t+1}) + q_{t+1}) \right]. \tag{8}
\]

With the caveat that agents use \( F^s \) instead of \( F^a \) to form expectations, these conditions are standard from models with credit constraints. Following Mendoza (2010), we can show that the

\( ^2 \)One could also specify a continuous AR(1) process for \( \kappa \) such as \( \kappa_t = m_t + \kappa_{t-1} + \epsilon_t \). The different regimes could be captured with a shift in the mean: \( m \in \{m^h, m^l\} \) and the agents could learn about the process governing \( m \). We conjecture that this specification would yield similar results as agents could turn optimistic about the persistence of the high mean regime for \( \kappa \).

\( ^3 \)In a more general case in which a similar informational friction affects also \( z \), the arguments of \( F^a \) and \( F^s \) would be the doubles \( (\kappa_{t+1}, z_{t+1}) \) and \( (\kappa_t, z_t) \).
second condition implies a forward solution for land prices in which the future stream of land
dividends is discounted at the stochastic discount factors adjusted for the shadow value of the
credit constraint:
\[ q_t = E^s_t \left[ \sum_{j=0}^{\infty} \left( \prod_{i=0}^{j} M_{t+i}^{l+1+i} \right) z_{t+1+j} g'(l_{t+1+j}) \right], \quad M_{t+i}^{l+1+i} = \frac{\beta(1 - \tau) u'(c_{t+1+i})}{u'(c_{t+i}) - \mu_{t+i} \kappa_{t+i}} \tag{9} \]

Defining the post-tax return on land as \( R^q_{t+1} = (1 - \tau)(z_{t+1} g'(l_{t+1}) + q_{t+1})/q_t \) and the period
marginal utility of consumption as \( \lambda_{t+1} = \beta u'(c_{t+1}) \), the post-tax excess return on equity can be
expressed as:
\[
E^s_t \left[ R^q_{t+1} - R \right] = \frac{(1 - \kappa_t) \mu_t - \text{cov}^s_t(\lambda_{t+1}, R^q_{t+1})}{E^s_t[\lambda_{t+1}]} \tag{10} \]

Thus, as in Mendoza (2010), the borrowing constraint enlarges the standard premium on land
holdings, driven by the covariance between marginal utility and asset returns, by introducing direct
and indirect effects. The direct effect is represented by the term \((1 - \kappa_t) \mu_t\). The indirect effect is
represented by the fact that the credit constraint hampers the agents’ ability to smooth consumption
and hence reduces \( \text{cov}^s_t(\lambda_{t+1}, R^q_{t+1}) \). Moreover, since the expected land returns satisfy \( q_tE^s_t[R^q_{t+1}] = (1 - \tau) E^s_t[z_{t+1} g'(l_{t+1}) + q_{t+1}] \), we can rewrite the forward solution for the agents’ land valuation
as:
\[
q_t = \sum_{j=0}^{\infty} \left( \prod_{i=0}^{j} \left( \frac{1}{E^s_t[R^q_{t+1+i}]/(1 - \tau)} \right) \right) z_{t+1+j} g'(l_{t+1+j}) \tag{11} \]

The expressions in (10) and (11) imply that the collateral constraint lowers land valuations because
it increases the rate of return at which future land dividends are discounted. Note also that land
valuations are reduced at \( t \) not just when the constraint binds at \( t \), which increases \( E^s_t[R^q_{t+1}] \), but
also if agents expect (given their beliefs \( F^s \)) that the constraint can bind at any future date, which
increases \( E^s_t[R^q_{t+1+i}] \) for any \( i > 0 \). Thus this expression suggests that the learning process and
the collateral constraint interact in an important way. For instance, suppose the credit constraint
was binding at \( t \), pessimistic beliefs such that agents expect higher future land returns because of
tight credit conditions will depress more current land prices, which will tighten more the collateral
constraint.

The economy has a fixed unit supply of land, hence market clearing in the land market implies
that the land holdings of the representative agent must satisfy \( l_t = 1 \) for all \( t \), and the rest of the
equilibrium conditions reduce to the following:

\[ u'(c_t) = \beta RE_t^s [u'(c_{t+1})] + \mu_t \]  \hspace{1cm} (12)

\[ q_t(u'(c_t) - \mu_t \kappa_t) = \beta (1 - \tau) E_t^s [u'(c_{t+1}) (z_{t+1}g'(1) + q_{t+1})] \]  \hspace{1cm} (13)

\[ c_t = z_t g(1) - \frac{b_{t+1}}{R} + b_t \]  \hspace{1cm} (14)

\[ \frac{b_{t+1}}{R} \geq -\kappa_t q_t 1 \]  \hspace{1cm} (15)

2.2 Learning Environment

Following Cogley and Sargent’s (2008), our learning setup features two-point passive learning without noise, so that \( p^s_t(\kappa_{t+1} | \kappa_t) \to p^a(\kappa_{t+1} | \kappa_t) \) for sufficiently large \( t \). With this setup, agents learn about the transition probability matrix only as they observe realizations of the shocks. In particular, they learn about the true regime-switching probabilities of \( \kappa \) only after observing a sufficiently long set of realizations of \( \kappa^h \) and \( \kappa^l \).

This learning setup fits nicely our goal of studying a situation in which financial innovation represents an initial condition with a state \( \kappa^h \) but with imperfect information about the true riskiness of this new environment. Agents are ignorant about the true transition distribution of \( \kappa \), since there is no data history to infer it from, and so they start with arbitrary priors. Over time, if they observe a sequence of realizations of \( \kappa^h \) for a few periods, they build optimism by assigning a probability to the possibility of transiting to the state \( \kappa^l \) that is lower than the true value. We refer to this situation as the “optimistic phase.” Such optimism by itself is a source of vulnerability, because it is quickly overturned with the first few realizations of \( \kappa^l \) that hit the economy. In addition, during the optimistic phase, the incentives to build precautionary savings against the risk of a shift in the ability to leverage are weaker than in the long-run RE equilibrium. This increases the agents’ risk of being caught “off-guard” (i.e. with too much debt) when the first shift to the low leverage regime occurs.

Modeling imperfect information in this fashion implies a deviation from rational expectations, but there is bounded rationality because agents use Bayes’ rule to update their beliefs about the transition probabilities. This is a key feature of our model, because it highlights the role of the short history of a new financial regime in hampering the ability of agents to correctly assess risk. This approach seems better suited to study the role of the U.S. financial innovation process in causing the financial crisis, as opposed to a standard signal extraction problem with noisy signals and rational
expectations. In our setup, agents observe variables perfectly, without noise. Their problem is that when the new regime starts, they do not know what are the true probabilistic processes driving risk, until sufficient data is available. Note also that time alone does not determine how fast agents learn about these processes. The sequence in which realizations and switches between realizations occur also matters.

The passive learning structure facilitates significantly the analysis and numerical solution of the model. In particular, it allows us to split the analysis into two parts. The first part uses Bayesian learning to generate the agents’ sequence of posterior density functions \( \{ f(F^s|\kappa^t) \}_{t=1}^T \). Each of these density functions (one for each date \( t \)) is a probability distribution over possible Markov transition matrices \( F^s \). Since agents do not know the true transition matrix \( F^a \), the density function changes with the sequence of realizations observed up to date \( t \) (i.e. \( \{ \kappa^t, \kappa^{t-1}, ..., \kappa^1 \} \) where \( \kappa^t = (\kappa_t, \kappa_{t-1}, ..., \kappa_1) \)) and with the initial date-0 prior, as we explain later. Notice that if date \( T \) is high enough, the sequence \( \{ f(F^s|\kappa^t) \}_{t=1}^T \) converges to a distribution with all its mass in \( F^a \). In other words, beliefs converge to the true values, so that in the long run the model converges to the RE equilibrium.

The second part of the analysis characterizes the model’s equilibrium by combining the sequences of posterior densities obtained in the first part, \( \{ f(F^s|\kappa^t) \}_{t=1}^T \), with chained sequential solutions from a set of “conditional” dynamic programming problems. These problems are conditional on the posterior density function of \( F^s \) that agents form with the history of realizations up to each date \( t \). This approach takes advantage of the fact that, since \( \kappa \) is exogenous, agents do not benefit from trying to improve their inference about the regime switching probabilities by “experimenting” using their optimization problems. As a result, the evolution of beliefs can be analyzed separately from the agents’ optimal consumption and savings plans. The remainder of this Section examines in more detail the Bayesian learning setup and the construction of the model’s equilibrium.

### 2.3 Learning and the Sequence of Beliefs

The learning framework takes as given an observed history of realizations of \( T \) periods of the exogenous shock, denoted by \( \kappa^T \), and a prior of \( F^s \) for date \( t = 0 \), \( p(F^s) \), and it yields a sequence of posteriors over \( F^s \) for every date \( t \), \( \{ f(F^s|\kappa^t) \}_{t=1}^T \). At every date \( t \), the information set of the agent includes \( \kappa^t \) as well as the possible values that \( \kappa \) can take \( (\kappa^h \text{ and } \kappa^l) \).

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4 In describing the learning problem, we follow the same notation as in Cogley and Sargent (2008).
Agents form posteriors from priors using a beta-binomial probability model. Since agents know the realization vector of \( \kappa \), imperfect information reduces to imperfect information only about the Markov transition matrix across \( \kappa' \)'s, and this in turn boils down to imperfect information about \( F_{hh}^a \) and \( F_{ll}^a \). The other two elements of the transition matrix of \( \kappa \) are recovered using \( F_{ii}^a + F_{ij}^a = 1 \), where \( F_{ij}^a \) denotes the probability of switching from state \( i \) to state \( j \).

The agents’ posteriors about \( F_{hh}^s \) and \( F_{ll}^s \) have Beta distributions as well, and the parameters that define them are determined by the number of regime switches observed in a particular history \( \kappa^t \). We assume that the priors for \( F_{hh}^s \) and \( F_{ll}^s \) included in \( p(F^s) \) are independent and determined by the number of transitions assumed to have been observed prior to date \( t = 1 \). More formally,

\[
p(F_{ii}^s) \propto (F_{ii}^s)^{n_{ii}^0 - 1}(1 - F_{ii}^s)^{n_{ij}^0 - 1}
\]

(16)

where \( n_{ij}^0 \) denotes the number of transitions from state \( i \) to state \( j \) assumed to have been observed prior to date 1.

The likelihood function of \( \kappa^t \) conditional on \( F_{hh}^s, F_{ll}^s \) is obtained by multiplying the densities of \( F_{hh}^s \) and \( F_{ll}^s \):

\[
f(\varepsilon^t | F_{hh}^s, F_{ll}^s) \propto (F_{hh}^s)^{(n_{hh}^t - n_{hh}^0)}(1 - F_{hh}^s)^{(n_{hl}^t - n_{hl}^0)}(1 - F_{ll}^s)^{(n_{lh}^t - n_{lh}^0)}(F_{ll}^s)^{(n_{ll}^t - n_{ll}^0)}.
\]

(17)

Multiplying the likelihood function by the priors delivers the posterior kernel:

\[
k(F^s | \varepsilon^t) \propto (F_{hh}^s)^{(n_{hh}^t - n_{hh}^0)}(1 - F_{hh}^s)^{(n_{hl}^t - n_{hl}^0)}(1 - F_{ll}^s)^{(n_{lh}^t - n_{lh}^0)}F_{ll}^s(n_{ll}^t - n_{ll}^0),
\]

(18)

and dividing the kernel using the normalizing constant \( M(\kappa^t) \) yields the posterior density:

\[
f(F^s | \kappa^t) = k(F^s | \kappa^t) / M(\kappa^t)
\]

(19)

where

\[
M(\kappa^t) = \int \int (F_{hh}^s)^{(n_{hh}^t - 1)}(1 - F_{hh}^s)^{(n_{lh}^t - 1)}(1 - F_{ll}^s)^{(n_{ll}^t - 1)}dF_{hh}^s dF_{ll}^s.
\]

The number of transitions across regimes is updated as follows:

\[
n_{ij}^{t+1} =
\]

\(^5\)This assumption of independent priors follows also Cogley and Sargent (2008), but it can be relaxed.
Note that the posteriors are of the form $F_{sh}^s \propto \Beta(n_{th}^{hh}, n_{t}^{hl})$ and $F_{bl}^s \propto \Beta(n_{th}^{lh}, n_{t}^{ll})$. That is, the posteriors for $\kappa^h$ only depend on $n_{th}^{hh}$ and $n_{t}^{hl}$ and not on the other two counters, $n_{th}^{lh}$ and $n_{t}^{ll}$, and the posteriors for $\kappa^l$ only depend on $n_{th}^{lh}$ and $n_{t}^{ll}$. This is important because it implies that the posteriors of $F_{sh}^s$ change only as $n_{t}^{hh}$ and $n_{t}^{hl}$ change, and this only happens when the date-$t$ realization is $\kappa^h$. If, for example, the economy experiences realizations $\kappa = \kappa^h$ for several periods, agents learn only about the persistence of the high leverage regime. They learn nothing about the persistence of the low leverage regime. To learn about this, they need to observe realizations of $\kappa^l$. Since in a two-point regime-switching setup persistence parameters also determine mean durations, it follows that both the persistence and the mean durations of leverage regimes can be learned only as the economy actually experiences $\kappa^l$ and $\kappa^h$.

We illustrate the learning dynamics of this setup by means of a simple numerical example. We choose a set of values for $F_{sh}^a$, $F_{bl}^a$, and initial priors, and then simulate the learning process for 300 quarters (75 years). The results are plotted in Figure 3. The true regime-switching probabilities are set to $F_{sh}^a = 0.95$ and $F_{bl}^a = 0.5$. These values are used only for illustration purposes (they are not calibrated to actual data as in the solution of the full model in Section 3). In addition, the initial priors are set to $F_{sh}^s \sim \Beta(0.7, 0.7)$ and $F_{bl}^s \sim \Beta(0.7, 0.7)$. With these priors the agents set their beliefs about the persistence of the high securitization regime to around 0.78 after observing the first period’s realization.

The most striking result evident in Figure 3 is that financial innovations, when “untested,” can lead to significant underestimation of risk. In other words, the initial sequence of realizations of $\kappa^h$ observed until just before the first realization of $\kappa^l$ (the “optimistic phase”) generates substantial optimism. In this example, this optimistic phase covers the first 30 periods. The degree of optimism produced during this phase can be measured by the difference between the conditional expectation based on the date-$t$ beliefs, $F_{sh}^s$, and the corresponding rational expectations value, $F_{sh}^a$ (horizontal line). As the Figure shows, this difference is much larger during the optimistic phase than in any of the subsequent periods. For example, even though the economy remains in $\kappa^h$ from around date 40 to date 80, the magnitude of the optimism that this period generates is smaller than in the initial optimistic phase. This is because it is only after observing at least once that a switch to $\kappa^l$ is possible that agents rule out the possibility of $\kappa^h$ being an absorbent state. As a result, $F_{sh}^s$ cannot surge as high as it did during the initial boom. Notice also that the first realizations of $\kappa^l$ generate a “pessimistic phase,” in which $F_{bl}^s$ is significantly higher than $F_{bl}^a$, so the period of overoptimistic expectations of high securitization is followed by a period of overpessimistic expectations.
Figure 3: Evolution of Beliefs

Notes: This figure plots the evolution of beliefs about $F_{hh}^s$ (top panel), $F_{ll}^s$ (middle panel), and the associated realizations of $\kappa$ (lower panel). The horizontal red lines in the upper two panels mark the true values of the corresponding variables.

Figure 3 also reflects the fact that the beliefs about the average duration of $\kappa^h$ ($\kappa^l$) are updated only when the economy is in the high (low) state. This is evident, for example, in the initial optimistic phase (the first 30 periods), when $F_{ll}^s$ does not change at all. As explained above, for the agents to learn about the duration of the high (low) securitization regime, the economy needs to actually be in that regime. This feature of the learning dynamics also explains why in this example $F_{hh}^s$ converges to $F_{hh}^a$ faster than $F_{ll}^s$. Given that the low securitization regime is visited much less frequently, it takes longer for the agents to learn about its persistence, and therefore $F_{ll}^s$ takes longer to converge to $F_{ll}^a$.

2.4 Recursive Equilibrium

We define the model’s competitive equilibrium in recursive form. The state variables of the representative agent’s recursive problem are the endogenous individual states $(b, l)$, the vector of aggregate states $X = [B, L, z]$ and $\kappa$. $B$ and $L$ are the aggregate states of bonds and land that the represen-
tative agent takes as given, along with their laws of motion \( B_{t+1} = H_B(X_t, \kappa_t), L_{t+1} = H_L(X_t, \kappa_t) \).

We will break down the problem into a set of conditional dynamic programming problems (CDPP), and add time indeces to the value and policy functions to identify the date of the CDPP to which they pertain. Consider first the date-1 problem. We pull \( f(F^s|\kappa^1) \) from the sequence of posterior density functions defined in the previous subsection. This is the first density function in the sequence \( \{ f(F^s|\kappa^t) \}_{t=1}^T \), and it represents the first posterior about the distribution of \( F^s \) that agents form, given that they have observed \( \kappa_1 \) and given their initial prior. Given \( H_B^B(X, \kappa) \) and \( H_L^L(X, \kappa) \) (i.e. the laws of motion of the date-1 problem), the representative agent solves the following date-1 CDPP:

\[
V_1(b_1, l_1, X_1, \kappa_1) = \max_{b_2, l_2} \left\{ u(c_1) + \beta \left[ \sum_{z_2 \in Z} \left( \int E^s[V_1(b_2, l_2, X_2, \kappa_2)|f(\kappa_2|\kappa^1, F^s)]f(F^s|\kappa^1)dF^s \right) \pi(z_2|z_1) \right] \right\}
\]

\[
F_1(b_1, l_1, X_1, \kappa_1) = \max_{b_2, l_2} \left\{ u(c_1) + \beta \left[ \sum_{z_2 \in Z} \left( \int E^s[V_1(b_2, l_2, X_2, \kappa_2)|f(\kappa_2|\kappa^1, F^s)]f(F^s|\kappa^1)dF^s \right) \pi(z_2|z_1) \right] \right\}
\]

where:

\[
E^s[V_1(b_2, l_2, X_2, \kappa_2)|f(\kappa_2|\kappa^1, F^s)] \equiv \sum_{n=1}^N \Pr(\kappa_2 = n|\kappa^1, F^s)V_1(b_2, l_2, X_2, \kappa_2)
\]

\( \Upsilon_1(X_1, \kappa_1) \) captures the lump sum government transfers. \( E^s[V_1(b_2, l_2, X_2, \kappa_2)|f(\kappa_2|\kappa^1, F^s)] \) is calculated in a similar way as the expectation in the right-hand-side of a standard Bellman equation for a conventional RE model with a discrete Markov chain of \( N \) realizations and transition function \( F^s \). Since agents in our model do not know \( F^0 \), however, the relevant expectation in the right-hand-side of (20) integrates over \( f(F^s|\kappa^1) \). The time subscripts that index the value functions in both sides of (20) indicate the date of the most recent observation of \( \kappa \) (which is date 1 in this case). It is also critical to note that, conditional on \( f(F^s|\kappa^1) \), this dynamic optimization problem remains recursive because the law of iterated expectations holds for Bayesian updating with passive learning (see Appendix B in Cogley and Sargent (2008)).

The solution to the date-1 CDPP is given by policy functions \( \hat{b}_2(b_1, l_1, X_1, \kappa_1) \) and \( \hat{l}_2(b_1, l_1, X_1, \kappa_1) \) and an associated value function \( \hat{V}_1(b_1, l_1, X_1, \kappa_1) \), both conditional on \( f(F^s|\kappa^1) \).

Generalizing the date-1 problem we can define the CDPPs for all subsequent dates \( t = 2, ..., T \)

\(^6\) As we explain later, our numerical solution method works using the model’s general equilibrium conditions, rather than Bellman equations, so that we can solve for the recursive equilibrium without solving directly for the aggregate laws of motion.
using the corresponding density function \( f(F^s|\kappa^t) \) for each date \( t \) in the sequence of posteriors solved for earlier. This is crucial because \( f(F^s|\kappa^t) \) changes as time passes, reflecting the passive Bayesian learning, which implies that the value and policy functions that solve each CDPP also change. This justifies using the time subscripts in the value and policy functions. Thus, in fact the “full solution” of the model is not recursive because history matters since different trajectories \( \kappa^t \) imply different densities \( f(F^s|\kappa^t) \), and hence different value and policy functions. If at any two dates \( t \) and \( t + j \) we give the representative agent the same values for \((b, l, X, \kappa)\), she in general, will not choose the same bond holdings for the following period because \( f(F^s|\kappa^t) \) and \( f(F^s|\kappa^{t+j}) \) will differ.

The date-\( t \) CDPP of the representative agent takes \( H^B_t(X, \kappa) \) and \( H^L_t(X, \kappa) \) as given and solves:

\[
V_t(b_t, l_t, X_t, \kappa_t) = \max_{b_{t+1}, l_{t+1}} \left\{ u(c_t) + \beta \sum_{s_{t+1} \in Z} \left( \int E^s[V_{t+1}(b_{t+1}, l_{t+1}, X_{t+1}, \kappa_{t+1})|\kappa_{t+1}, F^s]f(F^s|\kappa^t)dF^s \right) \pi(z_{t+1}|z_t) \right\}
\]

\[
s.t. \quad c_t = (z_t g(l_t) + q_t(X_t, \kappa_t)l_t)(1 - \tau) - q_t(X_t, \kappa_t)l_{t-1} - \frac{b_{t+1}}{R} + b_t + T_t(X_t, \kappa_t),
\]

\[
\frac{b_{t+1}}{R} \geq -\kappa q_t(X_t, \kappa_t)l_{t+1}
\]

where:

\[
E^s[V_{t+1}(b_{t+1}, l_{t+1}, X_{t+1}, \kappa_{t+1})|\kappa_{t+1}, F^s]] = \sum_{n=1}^N \Pr(\kappa_{t+1} = n|\kappa^t, F^s)V_t(b_{t+1}, l_{t+1}, X_{t+1}, \kappa_{t+1})
\]

The solution of this problem is characterized by policy functions \( \hat{b}_{t+1}(b_t, l_t, X_t, \kappa_t) \) and \( \hat{l}_{t+1}(b_t, l_t, X_t, \kappa_t) \) and an associated value function \( \hat{V}_t(b_t, l_t, X_t, \kappa_t) \), all conditional on \( f(F^s|\kappa^t) \), and hence conditional on the particular history \( \kappa^t \) that drives the posterior density of \( F^s \).

Given the recursive structure of each date-\( t \) CDPP, we can define the model’s recursive equilibrium as follows:

**Definition** Given a \( T \)-period history of realizations \((\kappa^T, \kappa^{T-1}, ..., \kappa^1)\), a recursive competitive equilibrium for the economy is given by a sequence of functions \([\hat{b}_{t+1}(b_t, l_t, X_t, \kappa_t), \hat{l}_{t+1}(b_t, l_t, X_t, \kappa_t), f(F^s|\kappa^t), H^B_t(X_t, \kappa_t), H^L_t(X_t, \kappa_t), q_t(X_t, \kappa_t), T_t(X_t, \kappa_t)]_{t=1}^T \) such that: (a) \( \hat{b}_{t+1}(\cdot), \hat{l}_{t+1}(\cdot) \) solve the date-\( t \) CDPP (21) conditional on \( f(F^s|\kappa^t) \) and given \( H^B_t(X_t, \kappa_t), H^L_t(X_t, \kappa_t); \) (b) \( f(F^s|\kappa^t) \) is the date-\( t \) posterior density of \( F^s \) determined by the Bayesian passive learning process summarized in Equation (19); (c) the decision rules and laws of motion satisfy the representative agent conditions \( \hat{b}_{t+1}(B_t, L_t, X_t, \kappa_t) = H^B_t(X_t, \kappa_t), \hat{l}_{t+1}(B_t, L_t, X_t, \kappa_t) = H^L_t(X_t, \kappa_t); \) (d) the market clearing condition in the land market holds \( H^L_t(X_t, \kappa_t) = L_t = 1 \); (e) government transfers satisfy
\( \Upsilon_t(X_t, \kappa_t) = \tau(z_t g(1) + q_t(X_t, \kappa_t)) \); and (f) \( q_t(X_t, \kappa_t) \) satisfies the asset pricing condition (11).

Intuitively, the complete solution of the recursive equilibrium is formed by chaining together the solutions for each date-t CDPP. That is, the equilibrium dynamics for a particular sequence of realized histories \( \kappa^T, \kappa^{T-1}, \ldots, \kappa^1 \) that agents observe at each date and use to update their beliefs are given by the sequences \( [H_t^B(X_t, \kappa_t), q_t(X_t, \kappa_t), f(F^s|\kappa_t)]_{t=1}^T \). At each date in this sequence, \( H_t^B(X_t, \kappa_t), q_t(X_t, \kappa_t) \) are the recursive functions that solve the corresponding date’s CDPP using \( f(F^s|\kappa_t) \) to form expectations. The sequence of equilibrium bond holdings that the model predicts for dates \( t = 2, \ldots, T + 1 \) would be obtained by chaining the relevant decision rules as follows: 
\[
b_2 = H_2^B(X_2, \kappa_2), \quad b_3 = H_3^B(X_2, \kappa_2), \ldots, \quad b_{T+1} = H_T^B(X_T, \kappa_T).
\]

### 2.5 Solution method

We solve the model by an Euler equation method that combines price and policy function iterations using the land pricing equation and the general equilibrium conditions (12)-(15). Note that by using the latter we can reduce the state space to \( (b, z, \kappa) \), since the resource constraint and the collateral constraint imply that at equilibrium \( c_t = z_t g(1) - \frac{b_{t+1}}{R} + b_t \) and \( \frac{b_{t+1}}{R} \geq -\kappa_t q_t \). By proceeding in this way we avoid using aggregate states and iterations to converge on the representative agent conditions, because we work off optimality conditions instead of the Bellman equation.

The algorithm proceeds in these steps:

1. Calculate a sequence of posteriors over \( F^s \) for every date \( t \), \( \{f(F^s|\kappa_t)\}_{t=1}^T \), for a given sequence of realizations of \( \kappa \).
2. Take the date-1 posterior \( f(F^s|\kappa_1) \) from the sequence in Step 1.
3. Conjecture a land pricing function \( q_1(b_t, z_t, \kappa_t) \) and solve for the policy function on bonds \( \hat{b}_1(b_t, z_t, \kappa_t) \) using the Euler equation (Equation(12)) the resource constraint (Equation(14)), the borrowing constraint (Equation(15)), and the posterior density function, \( f(F^s|\kappa_1) \), to evaluate \( E^s \).
4. Use the policy function from Step 2 and the asset pricing equation (11) to compute a new pricing function \( \hat{q}_1(b_t, z_t, \kappa_t) \). Note that we can use the current beliefs in computing this forward solution because the Law of Iterated Expectations still holds.

---

\(^7\)The land decision rule and law of motion are irrelevant at this point since at equilibrium they must always equal the fixed unit supply of land.
5. Compare $\hat{q}_1(b_t, z_t, \kappa_t)$ and $q_1(b_t, z_t, \kappa_t)$, if they satisfy a convergence criterion then $\hat{b}_1(b_t, z_t, \kappa_t)$ and $\hat{q}_1(b_t, z_t, \kappa_t)$ are the date-1 equilibrium functions. If not, construct a new guess of the pricing function using a Gauss-Seidel rule and return to Step 3.

6. Move to the date-2 posterior $f(F^s|\kappa^2)$ from Step 1 and return to Step 3 (now to solve for the date-2 equilibrium functions $\hat{b}_2(b_t, z_t, \kappa_t)$ and $\hat{q}_2(b_t, z_t, \kappa_t)$). Repeat for each date-$t$ posterior in the sequence $\{f(F^s|\kappa^t)\}_{t=1}^T$, solving each time for the corresponding date-$t$ equilibrium functions.

The following features of the recursive equilibrium highlight important implications of the passive Bayesian learning that are useful in implementing the above algorithm:

1. If $f(F^s|\kappa^{t+j}) = f(F^s|\kappa^t)$, the solutions for the date $t + j$ and the date $t$ equilibrium functions are the same—note that posteriors that satisfy this condition are possible as Figure 3 showed. This does not mean the equilibrium dynamics are the same. It only means that the value, policy and pricing functions are the same. As long as in a time series simulation of the equilibrium dynamics it is true that $(b_{t+j}, z_{t+j}, \kappa_{t+j})$ are not the same as $(b_t, z_t, \kappa_t)$, the actual values for $\hat{b}_{t+j+1}(b_{t+j}, z_{t+j}, \kappa_{t+j})$ and $\hat{q}_{t+j+1}(b_{t+j}, z_{t+j}, \kappa_{t+j})$ will differ from $\hat{b}_t(b_t, z_t, \kappa_t)$, $\hat{q}_t(b_t, z_t, \kappa_t)$.

2. The above suggests that, for a particular CDPP at some date $t + j$, we can speed convergence of the numerical solutions if, whenever $||f(F^s|\kappa^{t+j}) - f(F^s|\kappa^t)||$ is small enough under some metric, we use for the date $t + j$ problem initial guesses given by the date-$t$ solutions.

3. The solutions to each date-$t$ CDPP are not functionally related (i.e. solving a particular date-$t$ problem does not require knowing anything about the solution for any other date). Thus, the model can be solved by solving each date-$t$ problem independently.\(^8\)

4. If $j \leq T$ is large enough for $f(F^s|\kappa^{t+j})$ to converge to $F^a$ (for some converge criterion), the solutions for all dates $t \geq j$ collapse to a standard Bellman equation that solves the recursive RE equilibrium using the true Markov process $F^a$.

5. As explained earlier, the full equilibrium solution of the intertemporal sequence of allocations and prices from dates 0 to $T$ is obtained by chaining the solutions of each date-$t$ problem (for $t = 0, \ldots, T$), and this needs to be done for each different trajectory $(\kappa^T, \kappa^{T-1}, \ldots, \kappa^1)$ that

\(^8\)This fact can be used to develop a strategy to speed up the full solution different from the one suggested in (2), because in a computer with $x$ number of cores, we can solve $x$ problems for $x$ different dates at the same time.
one assumes to generate a sequence of posterior densities \( f(F^s|\kappa^1)|_t=1^T \). This suggests two solution strategies. One is to take a stance on a particular set \((\kappa^T, \kappa^{T-1}, ..., \kappa^1)\) based on actual realizations for a particular time period of actual data. The second is to generate a set of \(I\) trajectories \([(\kappa^T_i, \kappa^{T-1}_i, ..., \kappa^1_i)]_i=1^I\) using the true Markov process \(F^a\), solve the model for each, and then take averages across these different solutions at each date \(t\).

3 Quantitative Analysis

In this Section we study the model’s quantitative predictions for our financial innovation experiment. The experiment assumes that learning takes place over \(T\) periods. At \(t = 1\) financial innovation begins with the first realization of \(\kappa^h\), followed by an optimistic phase in which the same regime continues for \(J\) periods \(\kappa_t = \kappa^h\) for \(t = 1, ..., J\) and the arrival of the first realization of \(\kappa^l\) at date \(J + 1\) \(\kappa_{J+1} = \kappa^l\). After this, the economy remains in state \(\kappa^l\) from dates \(J + 1\) to \(T\). We start with a baseline scenario in which, for simplicity, the learning process only involves the securitization regime, because the Markov process of \(z\) is known and there are no interest rate shocks.

3.1 Calibration

The calibration requires setting parameter values for \(\beta, \sigma, R\), the Markov process for \(z\), the securitization regimes and the parameters of the learning setup \((\kappa^h, \kappa^l, \text{the initial priors } p(F^s), \text{and the length of the optimistic phase } J\)). We calibrate the model to a quarterly frequency at annualized rates using U.S. data. The calibration of the preference parameters, the interest rate, the TFP process and production function is fairly standard. The calibration of the learning parameters and the securitization regimes is set to match the U.S. credit boom of the period between the mid 1990s and the mid 2000s, as we explain below. Since the actual values of these parameters are more uncertain than the others, we conduct extensive sensitivity analysis to evaluate the robustness of our results to the assumptions in the baseline calibration.

We set \(\sigma = 2.0\), the standard value in quantitative DSGE models. The real interest rate is set to 2.66 percent annually, which is the ex-post average real interest rate on U.S. three-month T-bills during the period 1980Q1-1996Q4. The value of the discount factor is set to 0.91 which implies an annual discount rate of 0.1111. These figures correspond to a quarterly discount factor of 0.974 and a discount rate of 0.0266. We choose the value of \(\beta\) to match the standard deviation of
consumption relative to output in the data to that generated by the pre-innovation scenario with \( \kappa = \kappa^I \).

The Markov process for \( z \) is set to approximate an AR(1) process \( \ln(z_t) = \rho \ln(z_{t-1}) + \varepsilon_t \) fitted to U.S. HP-filtered real GDP per capita using data for the period 1965Q1-1996Q4. The estimation yields \( \rho = 0.869 \) and \( \sigma_e = 0.00833 \), which imply a standard deviation of TFP of \( \sigma_z = 1.6835 \) percent. We use Tauchen and Hussey’s (1991) quadrature method to construct the Markov approximation to this process assuming a vector of 9 realizations. The transition probability matrix and realization vector are reported in an Appendix available on request.

We normalize mean output to one, \( L = 1 \), and set the model’s parameters so that the resource constraint at the deterministic steady state (or at the average of the stochastic stationary state) matches the ratios of consumption and net credit market assets of households and non-profit organizations to GDP observed in U.S. data. The average ratio of household consumption expenditures (including nonperishables) to GDP is relatively stable in the data with a slight upward trend (it has a mean of 0.68 with a standard deviation of 0.019 over the 1985Q1-2008Q3 period). We take the 2007 value, which is 0.7052, and since we normalized mean output, consumption at the deterministic steady state is \( c = 0.7052 \).

Pinning down a proxy in the data for the long-run average of net credit market assets of the households is more difficult because, as Figure 1 shows, the ratio of these assets to GDP has been declining secularly for most of the last fifteen years. This is a measure of the large expansion in net credit that the U.S. households experienced (an unprecedented doubling from 35 percent of GDP in the mid 1990s, and on average since 1980, to more than 70 percent of GDP in 2008!). Since this ratio is non-stationary, we calibrate \( b \) to the most recent observation, which is for 2008 and is equal to -0.7136. Hence, at the deterministic steady state \( b = -0.7136 \).

Investment is not explicitly modelled and government expenditures are in the form of lump-sum rebates to households. To close the resource constraint we introduce a fixed, exogenous amount of autonomous spending \( A \). Thus, the normalized steady-state resource constraint is \( 1 = c + A - br/R \), and given \( b \) and \( c \) we calculate \( A \) as a residual \( A = 1 - c + br/R = 0.276 \).

The production function is Cobb-Douglas and displays decreasing returns to scale in land. We do not formally estimate the contribution of residential land to production however, we argue it is small. Hence we set \( \alpha \) to 0.1 and conduct sensitivity analysis. \( \tau \) is set such that the unconditional

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\(^{9}\)Consumption and GDP data are from the *International Financial Statistics* of the IMF. Net credit market assets of the households and non-profit organizations are from the Federal Reserve’s *Flow of Funds Accounts of the United States.*
Table 1: Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Discount factor (annualized)</td>
<td>0.91</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Risk aversion coefficient</td>
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</tr>
<tr>
<td>$b$</td>
<td>Net credit GDP ratio</td>
<td>-0.713</td>
</tr>
<tr>
<td>$c$</td>
<td>Consumption GDP ratio</td>
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</tr>
<tr>
<td>$A$</td>
<td>Lump-sum absorption</td>
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</tr>
<tr>
<td>$r$</td>
<td>Interest rate (annualized)</td>
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</tr>
<tr>
<td>$\rho$</td>
<td>Persistence of endowment shocks</td>
<td>0.869</td>
</tr>
<tr>
<td>$\sigma_e$</td>
<td>Standard deviation of TFP shocks</td>
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</tr>
<tr>
<td>$\alpha$</td>
<td>Factor share of land in production</td>
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</tr>
<tr>
<td>$L$</td>
<td>Supply of land</td>
<td>1.0</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Tax rate on income from production &amp; capital gains</td>
<td>0.13</td>
</tr>
<tr>
<td>$\kappa^h$</td>
<td>Value of $\kappa$ in the high securitization regime</td>
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</tr>
<tr>
<td>$\kappa^l$</td>
<td>Value of $\kappa$ in the low securitization regime</td>
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</tr>
<tr>
<td>$F_{hh}$</td>
<td>True persistence of $\kappa^h$</td>
<td>0.964</td>
</tr>
<tr>
<td>$F_{ll}$</td>
<td>True persistence of $\kappa^l$</td>
<td>0.964</td>
</tr>
<tr>
<td>$n_0^{hh}, n_0^{hl}$</td>
<td>Priors</td>
<td>0.1</td>
</tr>
</tbody>
</table>

long-run average of land value to output in a rational expectations version of our model with a constant $\kappa$ is 0.477 which is the average land value to output ratio in the data for 1980Q1-1996Q4. This implies $\tau = 0.13$ which is reasonable in the light of Mendoza, Rasin and Tesar (1994) findings.

We date the introduction of financial innovation and the first realization of $\kappa^h$ as of 1996Q1 (this is date 1 in our experiment). This is consistent with the observations that the New Community Reinvestment Act was passed in 1995, the first CDS was issued at JPMorgan as well as the first publicly available securitization of Community Reinvestment Act (CRA) loans took place in 1997, and that 1996 is the year in which the net credit assets-GDP ratio shown in Figure 1 started on its declining trend. We date the start of the financial crisis with the early stages of the subprime mortgage crisis at end 2006 as the net proportion of banks reporting tighter standards for mortgage loans jumped significantly to 16 percent in the first quarter of 2007, suggested by Figure 2. Hence, we assume that agents observe $\kappa^h$ through 2006Q4. This implies an optimistic phase of $J = 40$ quarters (10 years). We consider a learning period with a total length of $T = 48$ quarters, in which the first 40 realizations of $\kappa$ are $\kappa^h$ while the remaining 8 are $\kappa^l$.10

10These dates are broadly in line with those of Campbell and Hercowitz (2009) who study the welfare implications.
Next we calibrate the initial priors and the values of $\kappa^h$ and $\kappa^l$. We assume that $\kappa^l$ represents the level of securitization that the U.S. economy was able to support before the financial innovation of the 1990s. Hence, to calibrate $\kappa^l$ we first calculate the mean ratio of net credit market assets of the households to GDP in 1980-1996, which is -0.313. Then we set $\kappa^l$ such that the unconditional long-run average of bond holdings in a rational expectations version of our model with a constant $\kappa$ is -0.313. This exercise yields $\kappa^l = 0.645$.

We set the high securitization regime to $\kappa^h = 1.154$ in order achieve debt levels close to those in the data at the peak of optimism. In other words, the debt level in the data in 2006Q4 is around 70 percent and we set $\kappa^h$ in order to reach this level in period 40 in our forecast function exercise.

The beta-distribution parameters that determine initial priors are calibrated as follows. A natural choice for the priors of $F^n_{hh}$ and $F^n_{ll}$ would be $Beta(1,1)$, but this implicitly assumes that the agents have already observed one transition from $\kappa^h$ to $\kappa^l$ and one in the opposite direction. This is not desirable in our setting because we are assuming financial innovations for which the time series data is “too short” or non-existent. A $Beta(n^{ii}_0, n^{ij}_0)$ distribution requires $0 < n^{ii}_0$ and $0 < n^{ij}_0$. The lower those values (or the less the agents know about the innovation), the more optimistic agents turn about the persistence of the high securitization state after the first observation of $\kappa^h$, and remain more optimistic throughout the learning period. We set $F^n_{hh} \sim Beta(0.1, 0.1)$ and $F^n_{ll} \sim Beta(0.1, 0.1)$ and discuss the economic interpretation and implications of this calibration below. Reducing $n^{ii}_0$ and $n^{ij}_0$ further did not result in any significant change in the magnitude of over-borrowing or the size of the asset price boom generated by the model.

In order to further elaborate the economic meaning of choosing low values for $n^{ii}_0$ and $n^{ij}_0$, we plot the probability density functions for the Beta distribution with three different parameter sets in Figure 4. $Beta(0.1, 0.1)$ corresponds to our baseline calibration. Note that this parameter choice, the Beta distribution is U-shaped with almost all of the mass concentrated around 0 and 1. Given $n^{ii}_0 = n^{ij}_0$, this distribution is symmetric with a mean of 0.5 and a variance of 0.208. Similarly, $Beta(1,1)$ case also has a mean of 0.5 however, its variance is 0.129, almost half that of $Beta(0.1, 0.1)$.$^{11}$ Figure 4 also plots the probability density function of $Beta(40,1)$ in order to illustrate the beliefs, $F^n_{hh}$, in period 40. At this stage, the agents would have observed 40 transitions from high to high and only one transition from high to low state. With the observation of a long duration of the high state, this distribution is highly skewed to the right with all of the mass

$^{11}$Beta(1,1) is the same as a continuous uniform distribution, U(0,1).
Notes: This figure plots the probability density function of the Beta distribution with different \( n_{0i}^{ii} \) and \( n_{0i}^{ij} \) where \( Beta(n_{0i}^{ii}, n_{0i}^{ij}) \).

Intuitively, assuming priors of the form \( Beta(0.1, 0.1) \) for both \( F_{hh}^s \) and \( F_{ll}^s \) implies that the agents conjecture that there four likely scenarios: a) Both high and low securitization regimes are extremely persistent, i.e. \( F_{hh}^s \approx 1 \) and \( F_{ll}^s \approx 1 \), b) The high securitization regime is extremely persistent and the economy switches to the low securitization regime rarely and for a short time, i.e. \( F_{hh}^s \approx 1 \) and \( F_{ll}^s \approx 0 \), c) The low securitization regime is extremely persistent and high securitization regime occurs rarely and has a short duration, i.e. \( F_{hh}^s \approx 0 \) and \( F_{ll}^s \approx 1 \), d) Neither regime is persistent and the economy constantly transits between the two, i.e. \( F_{hh}^s \approx 0 \) and \( F_{ll}^s \approx 0 \).

After observing a few realizations of \( \kappa^h \), the agents realize the high persistence of the high securitization state and rule out the third and fourth scenarios. This is evident in Figure 5 where we plot the evolution of \( F_{hh}^s \) and \( F_{ll}^s \) assuming \( Beta(0.1, 0.1) \) and \( Beta(1, 1) \) priors. As discussed before, unless the economy switches to the low securitization state, the agents cannot learn about the persistence of that state. Compared to the evolution of \( F_{hh}^s \) under \( Beta(1, 1) \) priors plotted in the same figure, our baseline calibration with \( Beta(0.1, 0.1) \) priors leads the agents to turn optimistic right from the start. It is important to note that, unlike Cogley and Sargent (2008) who inject an initial pessimism to the agents’ priors, we do not build in an initial optimism. In our case, agents were not optimistic prior to period 1 because with \( Beta(0.1, 0.1) \) priors in period 0, \( F_{hh}^s = F_{ll}^s = 0.5 \) which is the mean of this Beta distribution as plotted in Figure 4. By leaving the
agents with the four plausible scenarios mentioned above, all we assume is that agents believe that either the switches in the securitization regimes will not be frequent (scenarios 1-3) or that there will be a switch every period (scenario 4).

At this point we have calibrated all of the parameters that are needed for solving the equilibrium with learning under our financial innovation experiment. Notice in particular that this does not require knowing the true transition probability matrix of $\kappa (F^a)$. Solving the CDPPs of the agents requires the sequence of beliefs about the transition probability matrix $(f(F^a|\epsilon^t)_{t=0}^T)$, which is determined with the parameters we already set. Still, calibrating the true transition probability matrix is necessary in order to evaluate the macroeconomic effects of the imperfect information friction by comparing against the standard RE solution. In particular, we will compare the dynamics of the optimistic phase with those of the RE model to quantify the magnitude of over-borrowing and the credit, consumption and asset price booms and busts.

We calibrate $F_{hh}^a$ in the true regime-switching process of $\kappa$ so that the mean duration of the high securitization regime is 7 years as reported in Mendoza and Terrones (2008). This implies $F_{hh}^a = 0.964$. With this calibration of $F_{hh}^a$ and conditional on observing $\kappa^h$ at date 1, the probability of observing $\kappa^h$ the following 39 periods is 0.241. We assume a symmetric process by setting $F_{lt}^a = 0.964$. 

Notes: Beta(0.1, 0.1) corresponds to our baseline calibration while Beta(1, 1) shows beliefs if the priors were uniformly distributed.
3.2 Quantitative Findings

Figure 6 plots “conjectured” ergodic distributions of bond holdings for dates 1, 5, 40, 41 and 48 in the learning model and the true ergodic distribution for the RE model. It is critical to note a key difference between these distributions: The ergodic distribution of the RE model represents the stochastic stationary state of the economy both under rational expectations and in the learning model (since in the long run agents learn the true regime-switching process $F^a$, and thus the long-run equilibrium is the same as under rational expectations). The “conjectured” ergodic distributions for the other dates in the learning model do not represent the model’s stochastic stationary state. Instead, they are the agents’ projections, or conjectures, of what the long-run equilibrium would look like if they forecast the dynamics of $F^a$ using their current beliefs (i.e. these distributions assume that the corresponding period’s beliefs about the transition probability matrix of $\kappa$ are correct).

This assumption is clearly not valid in the model’s equilibrium dynamics, because in the short run beliefs can change substantially from one period to another and deviate sharply from the true transition probability matrix. Plotting the conjectured and RE long-run distributions is useful, however, for illustrating the build-up of optimism during the optimistic phase of the learning model’s dynamics. Note in particular that bond holdings in the interval $[-0.69, -0.50]$ are never a long-run outcome in the RE distribution, but they would be projected to be using the agents’ date-40 beliefs as correct. It takes observing only the first realization of $\kappa^h$ for agents to turn overly optimistic with respect to rational expectations, and conjecture that long-run debt positions in the $[-0.60, -0.55]$ range are probable long-run equilibria, while in the RE distribution they have zero long-run probability. As optimism builds, the highest debt conjectured to have positive long-run probability of occurring rises, and the mass of probability assigned to debt levels larger than the largest debt under rational expectations also rises. This process peaks at the peak of the optimistic phase in date 40.

We study next the learning model’s equilibrium dynamics. We defined in the calibration a trajectory of realizations $\kappa^T$ with $T=48$, the first 40 are equal to $\kappa^h$ and the last 8 are equal to $\kappa^l$, and we used the calibrated values of the learning process to compute the sequence of beliefs $\left[f(F^s|\varepsilon^t)\right]_{t=1}^{48}$. We can then solve the corresponding CDPPs to obtain $\left[\hat{V}_t(b_t, \varepsilon_t), \hat{b}_{t+1}(b_t, \varepsilon_t)\right]_{t=1}^{48}$. Given these bond decision rules, we can determine the equilibrium bond dynamics by chaining the decision rules that correspond to each period’s beliefs. For dates $t = 1, ..., 40$, the equilibrium bond
Figure 6: Ergodic Distributions of Bond Holdings

Notes: This figure plots the ergodic distribution of bond holdings implied by the learning model in periods 1 (initial period), 5, 40 (peak of optimism), 41, and 48 as well that of the rational expectations model marked by “RE.”

dynamics are given by $b_2 = \hat{b}_1(b_1, \kappa^h, y)$, $b_3 = \hat{b}_2(b_2, \kappa^h, y)$, ..., $b_{40} = \hat{b}_{39}(b_{39}, \kappa^h, y)$, and those for dates $t = 41, ..., 48$ are given by $b_{41} = \hat{b}_{40}(b_{40}, \kappa^l, y)$, ..., $b_{48} = \hat{b}_{47}(b_{47}, \kappa^l, y)$. Notice we have solutions for this sequence for all initial conditions $b_1$ defined in the discrete grid of $b$ over which we solved the model, and for all values of $y$ in the discrete Markov process of output. Hence, in analyzing the dynamics we set $b_1 = -0.35$ and $y_1 = 1$, which aim to match observations from U.S. data for 1997q1, and study forecast functions to average across the exogenous output dynamics over time (full details are provided in the Appendix).

Figure 7 plots the forecast functions for bond holdings, consumption, price, and the saving flow as percent deviations from their long run means (which are the same as the long-run means of the RE scenario). The solid (blue) lines correspond to the learning model, the dashed (red) lines are for the fixed price scenario while the dotted (black) lines represent the RE model. The fixed price scenario is one with the price on the right hand side of the credit constraint set to a fixed value which in this case is the long run mean of the asset price:

$$\frac{b_{t+1}}{R} \geq -\kappa_t \bar{q}_{t+1}.$$
Notes: This figure plots the forecast functions of bond holdings to output ratio, price of land, consumption, and gross saving flow to output ratio (GSF/y) as percentage deviations from their long run means implied by the rational expectations scenario. GSF/y is calculated as \( (b'/R - b)/y \). Realizations of \( \kappa \) are as described in the text, \( \kappa^h \) in the first 40 periods and \( \kappa^l \) in the remaining 8. Bond holdings in period 1 are set to their values in 1997Q1 in the data. “Fixed q” refers to the scenario with the asset price on the right hand side of the credit constraint fixed at 0.459 which is the long run average of prices.

By doing so, in this scenario we shut down the debt-deflation channel in the same fashion with shutting down the informational friction when looked at RE. Note that even the RE and fixed price scenarios generate some dynamics in this exercise, because \( b_1 \) is not the mean of those scenarios and also because we are using a particular time-series of realizations of \( \kappa \) (instead of using the true Markov process of \( \kappa \) to build the forecast functions of \( \kappa \) conditional on \( \kappa_1 = \kappa^h \)).

Agents consistently borrow and consume more, and save less, than under rational expectations during most of the optimistic phase. Bond holdings as a share of output decline by as much as 35 percentage points below the long-run average, and then bounce back about 98 percentage points as optimism starts disappearing in period 41. These dynamics are in line with the downward trend
Table 2: Short-run Moments

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>RE</th>
<th>Fixed $q$</th>
<th>Baseline</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[(b/y)_{40} - (b/y)_0]$</td>
<td>-0.388</td>
<td>-0.155</td>
<td>-0.170</td>
<td>-0.351</td>
</tr>
<tr>
<td>$E[(ql/y)_{40} - (ql/y)_0]$</td>
<td>0.256</td>
<td>-0.029</td>
<td>0.120</td>
<td>0.086</td>
</tr>
</tbody>
</table>

Note: Data column reports the difference between 2006Q4 observation and the average of 1980-1996 for both variables. In columns 3-4 the realizations of $\kappa$ are set to the path described in the text, $\kappa^h$ in the first 40 periods and $\kappa^l$ in the remaining 8. Period 0 refers to pre-innovation period while period 40 is the peak of optimism. $qL/y$ is the aggregate market value of residential land divided by output.

In net credit market assets observed in the data.

In line with the debt dynamics, the forecast functions show low saving rates coupled with over-consumption (with respect to rational expectations) during the first 8 periods of the optimistic phase, followed by sharp corrections at date 41. The rise in consumption in the initial periods are financed by the sharp increase in bond holdings in those periods. After period 8, significant amount of debt accumulates leading to higher interest payments to the rest of the world and therefore lower consumption. In other words, as the debt builds, agents end up using the additional net borrowing to repay the interest on the existing debt rather than financing higher consumption.

The most important result illustrated in Table 2 is that the learning model generates significantly more debt in the short run than the RE or fixed price scenarios. Table 2 shows that the change in bond holdings from period 1 to 40 is about 20 percentage points of GDP lower in the learning setup than in the RE model and 18 percentage points lower than the fixed price scenario. Hence, financial innovation when agents are uncertain about the true nature of the new financial environment produces significant over-borrowing relative to what RE predicts.

Looking at the price dynamics, RE model leads to a fall in prices in absence of a change in beliefs throughout this experiment. On the contrary, the fixed price scenario generates a larger price boom than the baseline scenario. Since the size of the asset price boom reaches 8.6 percent of output at date 40 in our forecast functions, we can conclude that the informational friction accounts for about 1/3 of the 25 percentage point asset price boom observed in the data.

Next we investigate the excess returns dynamics to understand better the agents’ perception of the riskiness of land throughout the learning process. Figure 8 plots the one period ahead expected excess returns for 50 periods ahead of initial dates $t=1$, 40, and 41. In other words, for example period 40 shows the expectation of the agents looking into the future given beliefs and decision
rules as of period 40. These expectations vary depending on the state chosen for period \( t \). We set this state to the mean bond holdings in period \( t \) from the forecast function exercise to focus on the most “relevant” state in that period. Similarly, we set the values of \( \kappa \)s to their values in the corresponding period \( t \) in the forecast function exercise and set TFPs to their mean value.

Figure 8: Excess Returns

Notes: Expected excess returns for 50 periods ahead of initial dates \( t=1, 40, \) and \( 41 \), computed using the beliefs and associated equilibrium pricing function of the each date’s optimization problem. The expected returns are conditional on the bond holdings predicted for each initial date by the forecast functions of Figure 8, the mean value of TFP \( (\varepsilon = 1) \), and the value of kappa indicated in the history of realizations for each date \( t \).

Focusing on period 1 in Figure 8, the excess returns in RE exceed those of baseline learning setup in the first 23 periods. Staring from period 24, this ordering reverses. These dynamics imply that in period 1, the price is lower in RE than baseline learning. The fixed price scenario lies below both RE and baseline leading to a higher price in that scenario. This ordering of the different scenarios is consistent with the behavior of prices in period 1 as plotted in Figure 9.

As agents turn more optimistic after observing a sequence of high securitization regimes, they become willing to hold the risky asset with lower returns. This is evident in the top right panel
Notes: This figure plots asset prices for baseline learning, rational expectations and fixed price scenarios as percent deviations from the pre-innovation mean.

of Figure 8 that plots the expected excess returns for period 40. Baseline expected excess returns fall significantly below those of RE. And in period 41 when the switch to the low securitization takes place this ordering of baseline and RE reverses. The fixed price scenario in line with Figure 9 implies the lowest expected excess returns in period 40 with a significant increase in period 41.

4 Conclusion

In this paper we introduced informational frictions into a simple land model with an asset that has a fixed supply to study the implications of financial innovation in an environment in which agents do not know the true regime-switching probabilities across high and low debt securitization regimes. Agents are Bayesian learners, however, so in the long run, after observing a sufficiently long history of realizations of the financial regimes, they learn the true the transition probability matrix driving the Markov-switching process.

This imperfect information and learning structure is motivated by the fact that, when financial innovation emerged in the 1990s, financial markets participants did not have available a long time series of data to correctly calculate the true transition probability matrix across financial regimes. Since they were imperfectly informed about the true transition probabilities across regimes, and therefore about the true mean duration of a newly introduced high securitization regime, it is reasonable to argue that they would turn overly optimistic because their prior knowledge about the true riskiness of the new financial environment was limited, and the period between the mid
1990s and the mid 2000s was characterized by a very high ability to securitize debt. Despite the simplicity of the model, the informational friction and learning generate a substantial amount of over-borrowing, which accounts for 90 percent of the massive surge in net credit assets of U.S. households observed between 1997-2006. Moreover, the model also predicts a credit crunch, a collapse in consumption and a surge in private savings when the economy experiences the first realization of the low securitization regime and in the periods that follow.

Our work leads us to ponder an important tradeoff the financial innovation process. By definition, the true riskiness of brand-new financial products cannot be correctly evaluated when these products are first introduced. Hence, exposure to the credit boom-bust process we studied in this paper comes along with the potential benefits of financial innovation. Strong regulation and supervision of financial intermediaries in the early stages of financial innovation are therefore critical. Capital requirements can be used to contain the over-borrowing mechanism we studied in this paper, but on the other hand they have to be used carefully, because tight regulatory constraints on securitization introduce additional distortions on financial intermediation and can undermine all the benefits of financial innovation.
References


Appendix

Ergodic Distributions and Forecast functions under Rational Expectations

Define the date $t$ unconditional probability distribution over bonds, productivity and collateral coefficients as $\lambda_t(b, \varepsilon, \kappa)$. The law of motion that governs the evolution of this distribution is:

$$
\lambda_{t+1}(b', \varepsilon', \kappa') = \sum_{\varepsilon} \sum_{\kappa} \sum_{\{b:b'=g(b,\varepsilon,\kappa)\}} \lambda_t(b, \varepsilon, \kappa) \pi(\varepsilon'|\varepsilon)p(\kappa'|\kappa)
$$

where $g(b, \varepsilon, \kappa)$ is the policy function that sets the optimal decision rule for bonds, $\pi(\varepsilon'|\varepsilon)$ is the Markov transition probability for productivity shocks, and $p(\kappa'|\kappa)$ is the Markov transition probability for collateral coefficients (with the two Markov processes assumed to be independent). The unconditional limiting distribution of bonds, productivity and collateral coefficients is given by $\lambda(b, \varepsilon, \kappa)$, and it represents the fixed point of the above law of motion. Our algorithm computes the ergodic distribution exactly in this way, by performing iterations of the law of motion until $\lambda_t(b, \varepsilon, \kappa)$ and $\lambda_{t+1}(b', \varepsilon', \kappa')$ satisfy a convergence criterion.

Unconditional forecast functions are averages for the models endogenous variables computed at each date $t$ using the corresponding distribution $\lambda_t(b, \varepsilon, \kappa)$, starting from any initial condition $(b_0, \varepsilon_0, \kappa_0)$ with distribution $\lambda_0(b_0, \varepsilon_0, \kappa_0) = 1$. By construction, just like iterations on the above law of motion converge to the long-run distribution, unconditional forecast functions converge to unconditional long-run averages computed with the ergodic distribution, regardless of the initial conditions (as long as the ergodic distribution itself is unique and invariant, that is, as long as the fixed point of the law of motion is well defined).

Given $\lambda_t(b, \varepsilon, \kappa)$, the date $t$ conditional probability distribution over $\kappa_i$ for $i = h, l$ is defined as follows:

$$
\tilde{\lambda}_t(b, \varepsilon|\kappa_i) = \frac{\lambda_t(b, \varepsilon, \kappa_i)}{\sum_b \sum_{\varepsilon} \lambda_t(b, \varepsilon, \kappa_i)}
$$

Conditional forecast functions are averages for the models endogenous variables computed at each date $t$ using the corresponding distribution $\tilde{\lambda}_t(b, \varepsilon|\kappa_i)$. By construction, as $\lambda_t(b, \varepsilon, \kappa_i) \rightarrow \lambda(b, \varepsilon, \kappa)$, the date-$t$ conditional distribution $\tilde{\lambda}_t(b, \varepsilon|\kappa_i)$ converges to the corresponding long-run conditional distribution $\tilde{\lambda}(b, \varepsilon|\kappa_i)$. Moreover, conditional forecast functions of any endogenous variable converge to the corresponding conditional long-run average.
Approximate Forecast Functions in the Learning Model

The learning model has dynamics in the beliefs themselves, and hence the standard concepts of conditional and unconditional forecast functions do not apply directly. Intuitively, one can construct a set of these forecast functions and ergodic distributions at each date $t$ by using the corresponding date-$t$ beliefs to form all the expectations about future states. In light of this, we decided to look at what we called a forecast function in the learning model by averaging only over productivity shocks and tracking the decision rules produced at each date by the corresponding set of beliefs. Specifically, we did the following: Take as given $(b_0, \varepsilon_0, \kappa_0)$, then the relevant values of the “forecast function” of bonds in a learning period of length $T$ with a sequence of realizations $[\kappa_t]_{t=0}^T$ are:

$$\hat{b}_1 = E[b_1|(b_0, \varepsilon_0, \kappa_0), f(F^s|\kappa^0)] = h_0(b_0, \varepsilon_0, \kappa_0; f(F^s|\kappa^0))$$

$$\hat{b}_2 = E[b_2|b_0, f(F^s|\kappa^1)] = \sum_{\varepsilon^1} \sum_{\{b_1:b_2=h_1(b_1, \varepsilon_1, \kappa_1)\}} \pi(\varepsilon^1|\varepsilon_0)h_1(b_1, \varepsilon_1, \kappa_1; f(F^s|\kappa^1))$$

$$\hat{b}_3 = E[b_3|b_0, f(F^s|\kappa^2)] = \sum_{\varepsilon^2} \sum_{\{b_2:b_3=h_2(b_2, \varepsilon_2, \kappa_2)\}} \pi(\varepsilon^2|\varepsilon_0)h_2(b_2, \varepsilon_2, \kappa_2; f(F^s|\kappa^2))$$

$$\ldots$$

$$\hat{b}_{T+1} = E[b_{T+1}|b_0, f(F^s|\kappa^T)] = \sum_{\varepsilon^T} \sum_{\{b_T:b_{T+1}=h_T(b_T, \varepsilon_T, \kappa_T)\}} \pi(\varepsilon^T|\varepsilon_0)h_T(b_T, \varepsilon_T, \kappa_T; f(F^s|\kappa^T))$$

where $\pi(\varepsilon^t|\varepsilon_0) = \pi(\varepsilon_1|\varepsilon_{t-1})\pi(\varepsilon_{t-1}|\varepsilon_{t-2})\ldots\pi(\varepsilon_1|\varepsilon_0)$ is the probability of a particular history of realizations of productivity up to date $t$ (for $t \geq 0$), $[f(F^s|\kappa^t)]_{t=0}^T$ is the sequence of beliefs, and $h_t(b_t, \varepsilon_t, \kappa_t; f(F^s|\kappa^t))$ is the optimal decision rule for bonds determined by the date- $t$ dynamic programming problem using the date-$t$ beliefs and evaluated for the states $(b_t, \varepsilon_t, \kappa_t)$. Note that because of the recursive structure of the $\hat{b}_t$s in fact the expectations are conditional not just on date-0 states (i.e. $(b_0, \varepsilon_0, \kappa_0)$), but on the history of realizations $[\kappa_t]_{t=0}^T$ and the history of beliefs $[f(F^s|\kappa^t)]_{t=0}^T$. 

35
The equivalent objects under rational expectations are:

\[
\bar{b}_1 = E[b_1(b_0, \varepsilon_0, \kappa_0)] = g(b_0, \varepsilon_0, \kappa_0)
\]

\[
\bar{b}_2 = E[b_2|b_0] = \sum_{\varepsilon_1} \sum_{\{b_1: b_2 = g(b_1, \varepsilon_1, \kappa_1)\}} \pi(\varepsilon_1|\varepsilon_0)g(b_1, \varepsilon_1, \kappa_1)
\]

\[
\bar{b}_3 = E[b_3|b_0] = \sum_{\varepsilon_2} \sum_{\{b_2: b_3 = g(b_2, \varepsilon_2, \kappa_2)\}} \pi(\varepsilon_2|\varepsilon_0)g(b_2, \varepsilon_2, \kappa_2)
\]

......

\[
\bar{b}_{T+1} = E[b_{T+1}|b_0] = \sum_{\varepsilon_T} \sum_{\{b_T: b_{T+1} = g(b_T, \varepsilon_T, \kappa_T)\}} \pi(\varepsilon_T|\varepsilon_0)g(b_T, \varepsilon_T, \kappa_T)
\]

We can also express the forecast functions of the learning model with a slight modification of the standard treatment used under rational expectations. Define a law of motion for describing the evolution of the probabilities of pairs \((b, \varepsilon)\) for a given history of \(\kappa\),

\[
\chi_{t+1}(b', \varepsilon') = \sum_b \sum_{\varepsilon} \chi_t(b, \varepsilon) \pi(\varepsilon'|\varepsilon) I_t(b', b, \varepsilon, \kappa_t)
\]

where \(I_t(b', b, \varepsilon, \kappa_t)\) is a binary indicator such that \(I_t(b', b, \varepsilon, \kappa_t) = 1 \leftrightarrow b' = h_t (b, \varepsilon, \kappa_t; f(F^s|\kappa^t))\) and zero otherwise. At date-0, for example, we have \(\chi_0(b_0, \varepsilon_0) = 1\) for the particular initial conditions \((b_0, \varepsilon_0)\), and \(\chi_0(b, \varepsilon) = 0\) for all other pairs \((b, \varepsilon)\). We also have that \(I_0(b', b_0, \varepsilon_0, \kappa_0) = 1 \leftrightarrow b' = h_0 (b_0, \varepsilon_0, \kappa_0; f(F^s|\kappa^0))\) and zero for all other \(b'\), and we could add the indicators for all other possible initial conditions, but since this have \(\chi_0(b, \varepsilon) = 0\) they wash out from the probability law of motion. Hence we get \(\chi_1(h_0 (b_0, \varepsilon_0, \kappa_0; f(F^s|\kappa^0)), \varepsilon') = \pi(\varepsilon'|\varepsilon_0)\) for each \(\varepsilon'\) and zero otherwise (for all pairs \((b', \varepsilon')\) such that \(b' \neq h_0 (b_0, \varepsilon_0, \kappa_0; f(F^s|\kappa^0))\)). Now we can compute the expected bonds chosen at date 1 for beginning of period 2 as: \(\bar{b}_2 = E[b_2|b_0, f(F^s|\kappa^1)] = \sum_b \sum_{\varepsilon} \chi_1(b, \varepsilon_0) h_1 (b, \varepsilon, \kappa_1; f(F^s|\kappa^1))\).

At this point we can just add over the whole grid of bonds because the probabilities already have incorporated the information relevant for the right bond positions that the system can land on.

Alternatively, we can define the probability law of motion as:

\[
\chi_{t+1}(b', \varepsilon') = \sum_{\varepsilon} \sum_{\{b: b'=h_t(b, \varepsilon, \kappa_t; f(F^s|\kappa^t))\}} \chi_t(b, \varepsilon) \pi(\varepsilon'|\varepsilon)
\]

In writing it this way we take out the indicator function but keep track of only the relevant initial states that can land in each \(b'\) by constraining the set of \(b\) over which the summation is taken.
Expected Return Calculations

We choose an initial triple \((b_t, \varepsilon_t, \kappa_t)\) with initial bond holdings set to \(b_t = \hat{b}_t\). \(t\) is the period for which we would like to calculate the sequence of expected returns. \(\hat{b}_t\) stands for the mean bond holdings in period \(t\) based on the forecast functions as explained above. \(\varepsilon_t\) is set equal to 1. \(\kappa_t\) is set to its value used in the forecast function calculations for the corresponding period \(t\). We calculate expected returns for any date \(t + 1 + j\) as of date \(t\). This involves a numerator with the dividends and price of date \(t + 1 + j\) and a numerator with the price as of date \(t + j\), all of them being projected as of the initial date \(t\).

We proceed in two steps. First, we calculate the probability tree conditional on the initial triple \((b_t, \varepsilon_t, \kappa_t)\) up to \(J\) periods ahead. Second, we construct the \(E_t^s[R_{t+1+j}]\) sequence for \(j = 0, 1, ..., J\). Finally, as a cross check we recover the asset price in state \((b_t, \varepsilon_t, \kappa_t)\) of date \(t\), i.e. \(q_t(b_t, \varepsilon_t, \kappa_t)\), using the \(E_t^s \left( \frac{1}{E_{t+1}^s[R_{t+1+j}]} \right)\) sequence to recalculate price as the present discounted value of dividends discounted by expected returns.

In the first step, for \(j = 0\) we put all the mass to the initial state that we are conditioning our calculations on:

\[
\lambda_t^0(b_t, \varepsilon_t, \kappa_t) = 1.
\]

Going forward these distributions evolve according to:

\[
\lambda_{t+j+1}^j(b_{t+j+1}, \varepsilon_{t+j+1}, \kappa_{t+j+1}) = \sum_{\varepsilon_{t+j+1}} \sum_{\kappa_{t+j+1}} \sum_{b_{t+j+1} \in H_{t+j+1}} \lambda_t^j(b_{t+j}, \varepsilon_{t+j}, \kappa_{t+j}) \pi(\varepsilon_{t+j+1}|\varepsilon_{t+j}) p_t^s(\kappa_{t+j+1}|\kappa_{t+j})
\]

for \(j = 0, 1, ..., J\) where \(p_t^s\) stands for the subjective transition probability matrix of \(\kappa\) corresponding to period \(t\) and \(H_{t+j+1} = \{b_{t+j+1} : b_{t+j+1} = h_t(b_{t+j}, \varepsilon_{t+j}, \kappa_{t+j}|f(F_s|\kappa_t^t))\}\). The superscript \(t\) of \(\lambda_t^j\) highlights the fact that this is the date-\(t+j\) element for the law of motion that started with initial conditions \(\lambda_t^j(b_t, \varepsilon_t, \kappa_t)\) as of date \(t\), so that the probabilities are conditional on date \(t\).

To compute the expected returns we first take the date \(t + j\) element of the sequence of \(\lambda\)’s, \(\lambda_{t+j}^j(b_{t+j}, \varepsilon_{t+j}, \kappa_{t+j})\). Intuitively, this is the equilibrium probability of landing in a particular state \((b_{t+j}, \varepsilon_{t+j}, \kappa_{t+j})\) in \(t + j\) periods ahead of \(j\). Then we compute expected returns for any \(t + 1 + j\)
conditional on date \( t \) as:

\[
E_t^s[R_{t+1+j}^q] = \sum_{\varepsilon_{t+j+1}} \sum_{\kappa_{t+j+1}} \sum_{b_{t+j+1} \in H_{t+j+1}} \sum_{\varepsilon_{t+j}} \sum_{\kappa_{t+j}} \lambda_{t+j}^j(b_{t+j}, \varepsilon_{t+j}, \kappa_{t+j}) \pi(\varepsilon_{t+j+1} | \varepsilon_{t+j}) \\
\times p_t^s(\kappa_{t+j+1} | \kappa_{t+j}) \left[ q_t(b_{t+j+1}, \varepsilon_{t+j+1}, \kappa_{t+j+1}) + d(\varepsilon_{t+j+1}) \right] \frac{1}{(1 - \tau)} q_t(b_{t+j}, \varepsilon_{t+j})
\]

where \( d(\varepsilon) = z(\varepsilon)g'(l) \). Note that \( E_t^s[R_{t+1+j}^q] \) is in fact \( E_t^s[R_{t+1+j}^q](b_{t+j}, \varepsilon_{t+j}, \kappa_{t+j}) \). In other words, the one period ahead forecast of expected return depends on the state that the economy is in.

To confirm that in fact these calculations are correct, we recalculate \( q_t(b_t, \varepsilon_t, \kappa_t) \):

\[
q_t(b_t, \varepsilon_t, \kappa_t) = \sum_{j=0}^{J} \sum_{\varepsilon_{t+j+1}} \sum_{\kappa_{t+j+1}} \sum_{b_{t+j+1} \in H_{t+j+1}} \sum_{\varepsilon_{t+j}} \sum_{\kappa_{t+j}} \lambda_{t+j}^j(b_{t+j}, \varepsilon_{t+j}, \kappa_{t+j}) \pi(\varepsilon_{t+j+1} | \varepsilon_{t+j}) \\
\times p_t^s(\kappa_{t+j+1} | \kappa_{t+j}) \left( \prod_{i=0}^{j} \left( \frac{1}{E_t^s[R_{i+1+j}^q] / (1 - \tau)} \right) \right) d(\varepsilon_{t+j+1}).
\]

Summing over \( \kappa_{t+j+1} \) and multiplying by the transition probability matrix \( p_t^s(\kappa_{t+j+1} | \kappa_{t+j}) \) appear to be redundant however, the dependence of \( E_t^s[R_{t+1+j}^q] \) on all of the state variables justifies these operations.