Optimal Bank Regulation and Fiscal Capacity

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Abstract
Countries vary in their ability to bail out their banking sector in the event of a financial crisis. At the same time, financial regulation is synchronized across countries. This paper addresses the question of whether countries with different fiscal capacity should optimally have different ex-ante minimum bank capital requirements. In an environment with endogenously incomplete markets and overinvestment because of moral hazard and pecuniary externalities, I show that countries with larger fiscal capacity should have lower minimum bank capital requirements. This result is the opposite of what one may expect given that countries with larger fiscal capacity often have stronger moral hazard. I also show that countries with strong moral hazard should impose a minimum liquidity requirement, in addition the minimum capital requirement, and that the key result is robust to introducing costly equity.

1 Introduction
This paper addresses the question of whether countries with different abilities to bail-out their banking system during a liquidity financial crisis should have more or less stringent ex-ante minimum bank capital requirements.

A bank bail-out can be financed either by taxing or borrowing and countries vary in their ability to use these instruments. I refer to this capability as the fiscal capacity of a country. For example, during a crisis, a country with a larger banking sector relative to its GDP, such as Switzerland, will be able to bail-out a smaller fraction of its banking sector than a country such as the United States, which has a smaller banking sector relative to its GDP. This will be the case because, for a given bail-out amount, the marginal cost of distortionary taxation will be smaller in the United States than in Switzerland. Alternatively, all else being equal, a country with a larger sovereign debt will optimally provide a smaller bank bail-out than a country with a smaller government debt, given that its marginal cost of borrowing will be higher. During the ongoing European sovereign

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debt crisis, Greece and Spain are two prime examples of countries that are fiscally constrained due to high levels of government debt.

Despite clear differences in the ability of countries to provide bank bail-outs during financial crises, the Basel I Accords have set synchronized standards for the ex-ante minimum bank capital requirements across most countries. The minimum capital requirement constrains banks to finance at least a fraction of risky bank assets using equity. This fraction is referred to as the minimum bank capital ratio. In response to the 2007-2008 financial crisis, in 2011, Swiss regulators deviated from the Basel I norm of 8 percent minimum capital ratio by significantly increasing the minimum capital requirement to 19 percent. This triggered a vigorous debate as to whether the actions of Swiss regulators were optimal.

This paper provides a theoretical justification for heterogeneous cross-country minimum bank capital requirements, a hypothesis that contrasts with the current Basel standards. More specifically, I show that in a model with both pecuniary externalities and moral hazard, countries with larger fiscal capacity should have lower ex-ante minimum bank capital requirements.

Moral hazard is the standard justification for ex-ante bank regulation. When banks expect a bail-out in the future in the event of a crisis, they might overinvest today if they internalize the fact that their actions affect the size of the bail-out received. Since a bigger bail-out might lead to stronger moral hazard, intuitively, one would expect that larger fiscal capacity would imply higher (not lower) minimum bank capital requirements. The reason why, in the model developed in this paper, this intuition proves to be incorrect relies crucially on the fact that the ex-ante regulatory instrument considered, a minimum capital requirement, is a "quantity" policy instrument and on the assumption that the policy maker has a sufficient number of instruments to replicate the constrained Central Planner's allocation. By setting the minimum bank capital ratio, the policy maker can directly determine the amount of bank investment and the moral hazard does not play a role since it does not affect the optimal amount of investment chosen by the Central Planner.

Recently, pecuniary externalities have gained in popularity as an alternative mechanism that can lead to overinvestment. More specifically, financial institutions do not internalize the fact that the more they invest ex-ante, the larger the fire sale of financial assets during a future crisis is. Pecuniary externalities are the main driving force behind the key result of this paper. The intuition is that for a given level of bank assets, the fire sale of financial assets is smaller in countries that can afford a larger bail-out. As a result, countries with larger fiscal capacity, such as the United States, can 'afford' larger investment booms ex-ante, which is why they should optimally impose lower minimum bank capital requirements relative to countries with smaller fiscal capacity, such as Switzerland.

The World Bank survey on bank regulation indicates that, in 2010, the majority of countries had a minimum capital requirement of eight percent, which is the capital ratio recommended by Basel I. Among the high income OECD countries, 25 out of 27 had a minimum bank capital requirement of 8 percent (the exceptions are Israel and Estonia).
In what follows, I describe the model in more detail and provide intuition for the key results. The model has three periods — zero, one and two — and there is uncertainty only in the middle period, in which the state of nature can be either high or low. A crisis is defined as the low state of nature where there is a fire sale. There are two types of risk neutral agents — bankers, modelled as entrepreneurs (borrowers) and consumers (lenders). Consumers own the banks and receive all the profits in the form of dividends. In addition, there is a policy maker who can provide a targeted bail-out during a crisis and can also regulate the banks. In the model, the bail-out is costly and is financed via a tax on consumers during the crisis period. I assume that there is an exogenous deadweight loss of the aggregate bail-out, modelled as a convex cost. The convex cost function also includes a parameter that captures the fiscal capacity of the country. The smaller the fiscal capacity of a country is, the larger the marginal cost of the bail-out is.

Three key assumptions, which are fairly standard in the literature on endogenous market incompleteness and optimal regulation, are required to obtain the main result of this paper. (1) Bankers face a borrowing constraint; They cannot pledge the whole value of their collateral and future cash flows. For example, in the core model, I assume that bankers can borrow only against collateral in the spirit of [Kiyotaki & Moore, 1997]. (2) The banking sector is more productive than the outside sector (the consumers). (3) The government can circumvent the borrowing constraint of the bankers via its power to tax the consumers.

All bankers (entrepreneurs) have an access to a linear production technology. They invest in period zero by borrowing and using their endowments. In period one, the capital stock has to be refinanced to remain productive. Also, the scale of the project can be increased or decreased. Assumption (1) guarantees that markets are endogenously incomplete, because the borrowing constraint limits the resources the banker can transfer into the crisis state. As a result, bankers are forced to sell capital to consumers, who have a downward sloping demand for capital that generates a fire sale. Assumption (2), when combined with endogenous market incompleteness, generates pecuniary externalities and overinvestment as shown in [Lorenzoni, 2008]. [Lorenzoni, 2008]’s framework is extended to include assumption (3). As a result, in the crisis state, the government optimally provides a bail-out to the banks.

In the core model, bankers are infinitesimally small and the equity is given by their period zero net worth, which is exogenous and fixed. Since the optimal bail-out is a function of only aggregate variables, which bankers take as given, there is no moral hazard in this set up, despite the presence of an ex-post anticipated targeted bail-out (moral hazard will be introduced later into the model). However, I show that ex-ante regulation is still justifiable due to the presence of pecuniary externalities, which generate overinvestment. Every banker does not internalize the fact that the more she invests ex-ante, the larger the fire sale will be in the future. As a result, her actions tighten the budget constraints of the other bankers and further exacerbate the fire sale and the inefficient transfer of resources from the bankers— the more productive sector — to the
consumers — the less productive sector. In this framework, the policy maker can replicate the constrained Central Planner’s allocation using a single ex-ante instrument.

The question that arises is how should the minimum bank capital requirement vary across countries with different fiscal capacity. I show that more fiscally constrained countries should impose higher minimum bank capital requirements. For a given amount of ex-ante investment, the countries that can provide a smaller bail-out during a crisis will have a larger fire sale, and the transfer of capital from the more productive sector — bankers — to the less productive sector — consumers — will be larger. Therefore, the Central Planner in more fiscally constrained countries perceives ex-ante investment as less attractive, and he optimally chooses to invest less in period zero than does the Central Planner of a country with a larger fiscal capacity. Since the period zero investment chosen by the Central Planner and the optimal minimum capital ratio are inversely related, smaller fiscal capacity implies optimally higher minimum bank capital ratio.

Next, I consider how the key result of the paper changes with the introduction of moral hazard into the model. In the real world, the banking sector in most countries is fairly concentrated. Therefore, it is important to consider how the results of the model change if the number of banks is finite. The key difference is that when banks are no longer infinitesimally small, they partially internalize their impact on aggregate prices and the amount of fire-sold capital, which introduces moral hazard into the model. I show that countries with fewer banks and larger fiscal capacity have stronger moral hazard.

The moral hazard in the model presents itself in two different dimensions. Bankers internalize the fact that the more they invest ex-ante, the larger the fire sale will be during a crisis in the future and, hence, the larger the bail-out will be. As a result, the moral hazard leads to an alternative source of ex-ante overinvestment (in addition to the pecuniary externality). Moreover, if the moral hazard is strong enough, bankers might "transfer" too little liquidity to the crisis state (pledge too high of a payment in the crisis state) relative to what the Central Planner would optimally do. The intuition is that a larger promised payment in the crisis state leads to a more severe fire sale and, therefore, to a larger expected bail-out. As a result, countries with a larger fiscal capacity and fewer banks might require a minimum liquidity requirement, in addition to the minimum capital requirement, to replicate the constrained Central Planner’s allocation. More than one hundred countries worldwide currently have some form of a minimum liquidity requirement. Therefore, understanding what is the role for minimum liquidity requirements and how they should optimally vary across countries with different fiscal capacity is an important question.

Finally, the constrained Central Planner’s allocation is not affected by the number of banks which implies that it is not affected by the presence or the severity of moral hazard. Therefore,

\[3\] From the World Bank survey on bank regulation, in 2010, the cross country average percent of total assets that was held by the five largest banks was 68%.

\[4\] The World Bank survey on bank regulation shows that in 2010, 103 out of 127 countries had some form of a minimum ratio on liquid assets, such as a regulatory minimum ratio on liquid assets as a percentage of total balance sheet or deposit base.
conditional on the policy maker being able to replicate the constrained Central Planner’s allocation, the same argument, as in the core model, remains regarding why more fiscally constrained countries should optimally have higher ex-ante minimum capital requirements even in the presence of moral hazard.

Whether countries with larger fiscal capacity should regulate more or less ex-ante depends crucially on the type of ex-ante regulatory instrument — a "quantity" instrument or a "price" instrument. In this paper, the focus is on the minimum capital requirement, which is a "quantity" instrument, because it is the ex-ante regulatory instrument that is currently implemented worldwide. While, the "price" instrument such as a tax on ex-ante investment can also be used to address overinvestment, the way through which a "price" instrument and a "quantity" instrument affect the ex-ante investment chosen by the bankers is very different. Consider the case where the policy maker has sufficient instruments to replicate the constrained Central Planner’s allocation. The minimum capital requirement directly sets the amount of period zero investment. In contrast, an ex-ante tax on investment affects only the marginal cost of ex-ante investment as perceived by the banker and does not affect the marginal benefit, which is a function of the moral hazard. Larger fiscal capacity implies stronger moral hazard and larger perceived marginal benefit of period zero investment by the banker relative to the Central Planner. As a result, if moral hazard is present, the optimal tax on period zero investment for countries with larger fiscal capacity is actually higher.

Finally, I introduce costly equity in the model with a continuum of banks (hence no moral hazard). In the specifications described so far, in line with the view of other papers on endogenously incomplete markets, the amount of equity was assumed to be exogenous and fixed to the endowment of the banker. To relax this assumption, the model with a continuum of banks is extended to allow bankers to borrow against their cash flow as well (non collateralized borrowing). Before the cash flow was non – pledgeable since the banker could simply run away with it. In the last section of the paper, consumers will be endowed with a costly monitoring technology, which, when used, will enable them to prevent the banker from running away with the cash flow. Since this form of non collateralized borrowing is costlier than collateralized borrowing, which is one of the key properties of equity, costly borrowing against the cash flow will be considered equivalent to raising equity.

Following are the new results that emerge in the model with costly equity. First, the presence of pecuniary externalities generates very different implications for the type of optimal bank regulation over the business cycle. As already argued, during non-crisis periods when there is no fire sale of financial assets, future pecuniary externalities and fire sales generate overinvestment that can be addressed using a minimum bank capital requirement. In contrast, during a crisis, contemporaneous pecuniary externalities might lead to underinvestment since bankers do not internalize the fact that the more they invest, the smaller the fire sale is. As a result, due to this underinvestment force, during a crisis, bankers might raise less equity than the amount of equity the constrained Central

\[\text{\textsuperscript{5}}\text{"Price" instruments, such as a tax on period zero capital, have been considered in the theoretical literature (see [Stein, 2012], [Bianchi, 2011] and [Jeanne & Korinek, 2012]).}\]

\[\text{\textsuperscript{6}}\text{This result is in the spirit of [He & Kondor, 2012], who show that pecuniary externalities can lead to underinvestment during crises and overinvestment during booms.}\]
Planner would want them to raise. This result provides a new theoretical justification to why it was optimal, during the 2007-2008 crisis, for US regulators to require from all banks to raise a minimum amount of equity.  

The second result that emerges from introducing non-collateralized costly borrowing is that when the only ex-ante instrument used is the minimum bank capital requirement, the constrained Central Planner’s allocation can no longer be replicated. The minimum bank capital requirement no longer pins down the ex-ante investment of the bankers. As a result, the bankers are tempted to circumvent it by raising more costly equity than what they would optimally raise if there was no minimum capital requirement so that they can overinvest relative to the investment by the constrained Central Planner. To replicate the constrained Central Planner’s allocation, a second instrument, such as a maximum amount of period zero investment, will be required. Given that such an instrument is not employed in practice and that the focus of this paper is on currently used regulatory instruments, it is worthwhile to ask the following question: What should be the optimal minimum capital requirement in the absence of any other ex-ante policy instruments? Using simulations I show that the result that countries with smaller fiscal capacity should optimally have larger minimum capital requirements remains.

**Related Literature**

The three key assumptions that led to the main result of this paper are fairly standard in the literature. The first assumption that bankers face a borrowing constraint is in the spirit of [Kiyotaki & Moore, 1997] and [Hart & Moore, 1994]. The second assumption, namely the banking sector is more productive than the outside sector (the consumers), has been used in many papers such as [Kiyotaki & Moore, 1997]. It can be further justified by the fact that financial institutions are considered more efficient than savers at providing credit to firms, because of their ability to monitor the borrower at a lower cost (for example see [Holmstrom & Tirole, 1997]). Hence, the value of loans (capital) is higher in the hands of the bankers than in the hands of the consumers. On the empirical side, the importance of the banking sector and the value lost by terminating the relationship between banks and firms are studied by the literature on relationship banking (see [Freixas & Rochet, 2008] for a literature review). The third key assumption that the government can circumvent the constraints of borrowers via its power to tax was used, besides others, by [Holmstrom & Tirole, 1998] and [Gorton & Huang, 2004]. Similarly to this paper, they also show that such government intervention can be welfare improving. 

Starting with the seminal work of [Bagehot, 1873], moral hazard has been proposed as one of the main reasons for bank regulation. However, a growing literature on financial sector regulation has

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7The justification suggested in the literature for this policy is related to the seminal paper on asymmetric information and costly equity by [Myers & Majluf, 1984], and it differs from the mechanism proposed in this paper. The [Myers & Majluf, 1984] story is the following. During the crisis, all banks were forced to raise equity at the same time, which improved the pool of banks that raised equity and alleviated the asymmetric information problem. As a result, the cost of raising equity decreased which was welfare improving.

8Other papers that find ex-post bail-outs to be welfare-improving, for example, are [Bianchi, 2012] and [Keister, 2012].
emerged, which emphasizes the role of fire sales and pecuniary externalities — [Lorenzoni, 2008], [Stein, 2012], [Jeanne & Korinek, 2012]. The importance of fire sales during financial crises has been emphasized by many papers starting with the seminal work by [Shleifer & Vishny, 1992] (see [Shleifer & Vishny, 2011] for a survey of the literature). I build on the paper by [Lorenzoni, 2008] who shows how pecuniary externalities can emerge in a microfounded environment \(^9\). The difference between this paper and the one by [Lorenzoni, 2008] is that neither he allows for an ex-post bailout, nor does he ask the question how optimal policy should vary with the fiscal capacity of the country, which is the key contribution of this paper. The only other paper, besides this one, that studies the optimal mix of ex-ante regulation and ex-post bail-out which has both pecuniary externalities and moral hazard is the one by [Jeanne & Korinek, 2012]. In contrast to this paper, in [Jeanne & Korinek, 2012] markets are exogenously incomplete. Endogenous market incompleteness is important to understand the key sources of inefficiency. It is also crucial for some of the key results of the paper such as the result that, for countries with stronger moral hazard, there is a role for a minimum liquidity requirement. Most importantly, [Jeanne & Korinek, 2012] do not ask the question raised by this paper: How the optimal mix of ex-ante and ex-post bank regulation should vary with the fiscal capacity of the country?

Some of the most recent papers which also find that there is a role for minimum liquidity regulations due to moral hazard are [Farhi & Tirole, 2012] and [Keister, 2012]. In an environment with a non-targeted bailout in the form of lowering the borrowing rate of banks, [Farhi & Tirole, 2012] show that there are complementarities in the actions of bankers and, as a result, multiple equilibria. If banks expect low interest rates during crises, they might end up holding too little of the safe asset which, in equilibrium, will force the policy maker to keep interest rates low. As a result, minimum liquidity requirements can improve welfare. In a [Diamond & Dybvig, 1983] environment with multiple equilibria, [Keister, 2012] shows that an expected government bail-out ex-post leads to moral hazard and to bankers choosing lower liquidity ex-ante relative to what the Central Planner would choose.

The Basel Accord recommendation of synchronized regulation is often justified by the idea of creating a "level-playing" field for banks. For a summary of the bank regulation literature see [Santos, 2001]. To my knowledge, there is only one other theoretical paper that argues in favor of heterogeneous cross country bank regulation. [Acharya, 2002] studies whether minimum capital requirements should be synchronized across countries given the presence of different bank closure policies. He finds that ex-ante synchronization of bank regulation is not optimal.

The structure of the paper is the following. In sections 2 to 5, I present the core model with

\(^9\) There is a large literature on pecuniary externalities where the inefficiency comes from binding borrowing constraints and prices enter the borrowing constraint (for example [Stein, 2012], [Bianchi, 2011]). In this paper, as in [Lorenzoni, 2008], the source of the pecuniary externality will be that bankers do not internalize the fact that their actions are tightening the *budget* constraints of the other bankers, not the *borrowing* constraints.

\(^{10}\) [Lorenzoni, 2008] also assumes that in addition to the borrowing constraint on the side of the bankers there is a limited commitment friction on the side of the consumers. I show that as long as the banks are owned by the consumers (an implicit ex-post transfer), limited commitment of the consumers will not be necessary to generate pecuniary externalities.
pecuniary externalities and no moral hazard. Section 6 considers the case of a finite number of banks which introduces moral hazard, in addition to the pecuniary externality. Section 7 compares "quantity" regulatory instruments to "price" regulatory instruments. In section 8, I allow the bankers to raise costly equity. Section 9 concludes and provides further discussion.

2 Model Set-Up

This section presents the set up of the core model without moral hazard. There are three agents in the economy — consumers, bankers/entrepreneurs and the government. The banks are owned by the consumers and the consumers receive all the profits in the form of dividends.\textsuperscript{11} Hence, all the dividends are paid out to the consumers who are risk neutral. The model is a three period model where $t = 0, 1, 2$ and there is no discounting. There is uncertainty only in the middle period, $t = 1$. In $t = 1$ there are two states of nature — a high state and a low state. The period zero probability of the high state occurring is $h$ and the period zero probability of the low state occurring is $l$ where $l + h = 1$. In equilibrium, the fire sale will occur only in the low state in period one, which is what I will refer to as the crisis state. There are two goods — a capital good and a consumption good where the consumption good is the numeraire good. Each period and state of nature, the consumption good can be transformed one to one into a capital good. There is no storage technology either for the capital good or for the consumption good. The capital good has to be employed in a production technology in every period and the consumption good is perishable.

2.1 Bankers

Assume that there is a continuum of bankers/entrepreneurs distributed uniformly over the unit interval.\textsuperscript{12} Every banker has access to a bank specific production technology. In $t = 0$, banker $i$ has to choose the amount he invests given by $k_0^i$. In $t = 1$ and state $s$, the project produces $a_1s k_0^i$ units of the consumption good where $a_{1h} < a_{1i}$. In $t = 1$, in order for the capital stock, $k_0^i$, to remain productive, it has to be refinanced. The banker has to invest an additional amount of $\gamma < 1$ per every unit of period zero capital, $\gamma k_0^i$ in total. Otherwise, the capital depreciates one hundred percent. I assume that the resale price of capital in period one, $q_{1s}$, is always great than $\gamma$ and, hence, all the capital is refinanced. In addition, in $t = 1$, banker $i$ also decides on the period one scale of the project, $k_{1s}^i$. If the banker has the resources and finds it optimal, he can increase the scale of the project, $k_{1s}^i > k_0^i$. If he does not have the resources or it is not optimal to do so, he can end up investing less than the period zero investment, $k_{1s}^i < k_0^i$. The capital sold by banker $i$ in

\textsuperscript{11}This is equivalent to assuming that the policy maker maximizes a weighted average of the consumers’ and bankers’ welfare where the weight on the bankers’ utility is zero. The model can be easily changed so that some weight is placed on the bankers as well or I can impose a requirement for an ex-ante welfare Pareto improvement for both the bankers and the consumers. The results remain unchanged as long as the government has an access to ex-post lump sum transfers from the banker to the consumer.

\textsuperscript{12}This assumption will be relaxed later, when I consider the case of a finite number of banks.
period one is \( k_{1s}^T = \min \{ 0, k_0^i - k_{1s}^i \} \) I will refer to \( k_{1s}^i \) as the "fire-sold" capital and the aggregate amount of fire-sold capital is \( k_{1s}^T = \int_0^t k_{1s}^i \, dt. \) In equilibrium, there will be a fire sale only in the crisis state. Finally, in \( t = 2, \) there is no further uncertainty and the amount invested in \( t = 1 \) produces \( A k_{1s}^i, \) where \( A > 0, \) and the capital can be sold to the consumers for the price of \( q_{2s}.\)

In \( t = 0, \) each banker is endowed with \( n \) units of the consumption good. Banker \( i \) can also borrow from the consumers but credit markets are imperfect due to an agency friction in the spirit of [Hart & Moore, 1994] and [Kiyotaki & Moore, 1997]. The contract that will emerge in equilibrium is a short term state contingent contract subject to a collateral constraint on the part of the banker. In \( t = 0, \) banker \( i \) can sell a promise to pay \( d_{1s}^i \) units of the consumption good in \( t = 1, \) state \( s, \) at the price \( \pi_s p_{1s}. \) This implies that the total period zero borrowing of banker \( i \) is \( \sum_s \pi_s p_{1s} d_{1s}^i. \) Also in \( t = 1 \) state \( s, \) banker \( i \) can sell a new promise to pay \( d_{2s}^i \) units of the consumption good in \( t = 2 \) state \( s \) at the price \( p_{2s}. \) As a result, his period one, state \( s \) borrowing is \( d_{2s}^i p_{2s}. \) The prices, \( p_{1s} \) and \( p_{2s}, \) will be determined in equilibrium. The amount that the banker can borrow is limited by the state contingent value of his collateral; \( d_{1s}^i \leq \theta (q_{1s} - \gamma) k_0^i \) and \( d_{2s}^i \leq \theta q_{2s} k_{1s}^i \) \( 1 - \theta \) is the fraction of the value of the collateral that has to be paid in legal fees if the consumer has to seize the collateral.

More specifically, the reason why the optimal contract that emerges in equilibrium is a short term state contingent contract subject to a collateral constraint is the following. I assume that the state of nature is verifiable and contractible but the banker can always run away with the cash flow in \( t = 1, a_{1s}, \) and in \( t = 2, A. \) However the consumer can seize the collateral and resell it. In period zero, banker \( i \) can borrow only against the value of period one collateral. The reason is that once in period one, banker \( i \) gives an enforceable "take-it-or-leave-it" offer to the consumer — either take \( \theta (q_{1s} - \gamma) k_0^i \) in period one as a payment or the bank will be closed (the banker will withdraw his human capital) and no output will be produced in period two. If the bank is closed the consumer will withhold the capital, pay the legal fees and the refinancing cost and resell the capital, which will generate \( \theta (q_{1s} - \gamma) k_0^i \) units of the consumption good. In equilibrium, the consumer will always accept the "take-it-or-leave-it" offer. Expecting that, in \( t = 0, \) the consumer will never lend against the value of period two collateral of the banker. Once the banker repays his old debt in \( t = 1, \) he can go to the credit market and enter a new collateralized contract. Since, in period two, he will never pay more than the value of the collateral after legal fees, the maximum amount he can borrow in \( t = 1 \) and state \( s \) is \( \theta q_{2s} k_{1s}^i. \)

Finally, banker \( i \) might receive an additional source of funding in the crisis state in the form of a bail-out. In the low state in period one, banker \( i \) receives \( B_{1l}^i \) as a transfer from the government, where \( B_{1l}^i \) is endogenously determined. Banker \( i \) can pay dividends every period and state of nature but he will optimally do so only in the last period, \( t = 2, \) when he gives all of the profits to the consumers.

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For simplicity, I assume that in the last period the refinancing cost is zero. This assumption can be relaxed without changing the result.
2.2 Consumers

There is a continuum of risk neutral consumers distributed uniformly over the unit interval. The preferences of the representative consumer are given by

\[ U_c^0 = c_0 + \sum_s \pi_s (c_{1s} + c_{2s}) \]

In every period and state of nature, the consumers receive an endowment \( e \). They can enter a state contingent contract with each banker both in periods zero and one as described in the previous section.

Consumers also have an access to a production technology which uses capital as an input good and transforms it into the consumption good within the same period. Once the production technology produces the consumption good, the capital depreciates one hundred percent. In \( t = 0 \) and \( t = 2 \), assume that the production technology is such that one unit of capital is transformed into one unit of consumption. Hence the price of capital in \( t = 0 \) and \( t = 2 \) will be pinned down to one, \( q_0 = q_2s = 1 \). When modelling the production technology of the consumers in \( t = 1 \), in order to generate a downward sloping demand for capital, I use an approach similar to many papers in the literature on financial frictions that follow the seminal paper of [Kiyotaki & Moore, 1997]. In \( t = 1 \), the production technology of the consumers is given by \( F(k_{1s}^T) \) where \( k_{1s}^T \) is the amount of capital employed which will turn out to be the aggregate amount of capital fire-sold by the bankers.\(^{14}\) In equilibrium, if the production technology of the consumer is employed, the price of capital will be pinned down by the marginal product of capital, \( q_{1s} = F'(k_{1s}^T) \).\(^{15}\)

The production technology satisfies the following assumptions: \( F(\cdot) \) is continuous and at least twice differentiable; \( F'(0) = 1, F''(\cdot) < 0 \) and \( \lim_{k_{1s}^T \to \infty} F'(k_{1s}^T) \geq \gamma \). Assuming that \( F'(0) = 1 \) and \( F''(\cdot) < 0 \) implies that the production technology of the consumers will not be used unless there is a fire sale. Otherwise, it will never be profitable for the consumers to transform consumption one to one into capital in \( t = 1 \) in order to use the production technology since the marginal product of capital is less than one if the technology is employed. The assumption \( F''(\cdot) < 0 \) guarantees that the larger the fire sale is, the lower the price of capital will be, which is a proxy for a downward sloping demand for capital. Finally \( \lim_{k_{1s}^T \to \infty} F'(k_{1s}^T) \geq \gamma \) ensures that the scrapping of capital will never be optimal. Notice that the bankers’ production technology is more productive than the consumers’. A detailed solution to the consumer problem is provided in the Appendix, Section 10.1. Since bankers are risk neutral, in equilibrium, from the Euler equation of the consumer, the prices of the state contingent debt contracts will be \( p_{1s} = p_{2s} = 1 \).

\(^{14}\)The assumption that the production technology is different across periods is for simplification and can be relaxed. However, given that the model is finite periods, relaxing it would imply that there will be always fire sales in \( t = 2 \) both in the high and the low state, which will be an unappealing feature of the model.

\(^{15}\)The assumption that capital cannot be stored implies that consumers cannot simply purchase capital in \( t = 1 \) in the low state and keep it until \( t = 2 \) in order to use their production technology that allows them to transfer capital into consumption one to one. If that were the case there would be no fire sales.
2.3 Government

The policy maker optimizes the ex-ante welfare of consumers who are the owners of the banks. He has an access to an ex-ante and an ex-post policy instrument. The ex-ante policy instrument is a minimum bank capital requirement defined as \( \rho^i \leq \frac{k_i}{\bar{x}_i} \), where \( \rho^i \) is the minimum bank capital ratio of bank \( i \). \( \rho^i \) is the minimum fraction of period zero investment that has to be financed using equity. The ex-post regulatory instrument is an optimal government bail-out during crises. I implicitly assume that the government can circumvent the collateral constraint of bankers during crises via its power to tax the consumers and transfer resources to the bankers. For simplicity, I assume that bail-outs are prohibitively costly when there is no crisis due to high political costs of transferring money from taxpayers to the financial system.\(^{16}\)

The bail-out is financed by taxing the consumers in the crisis state, where the tax levied is \( T_l \). I assume that bailing out the banking system is costly, and this cost can vary across countries. The size of the deadweight loss from the bail-out is given by \( \delta(B_l, \chi) \) where \( B_l = \int_0^1 B_l' d\bar{x} \) is the aggregate bail-out\(^{17}\), \( B_l' \) is the bail-out given to bank \( i \), and \( \chi \) is an exogenous parameter that captures how fiscally constrained the country is. The larger \( \chi \) is, the more fiscally constrained the country is (lower fiscal capacity). Market clearing implies that \( T_l = B_l + \delta(B_l, \chi) \).

I assume that \( \delta(B_l, \chi) \) is a convex and increasing function with respect to the aggregate bail-out, which implies \( \frac{\partial \delta(B_l, \chi)}{\partial B_l} > 0 \) and \( \frac{\partial \delta(B_l, \chi)}{\partial \chi} > 0 \). \( \frac{\partial \delta(B_l, \chi)}{\partial B_l} > 0 \) guarantees that the larger the total size of the bail-out is, the larger the deadweight loss is. \( \frac{\partial \delta(B_l, \chi)}{\partial B_l} \) is the marginal cost of the bail out increases with the total size of the bail out. The convexity of the deadweight loss is a standard assumption in the public finance literature to capture the cost of distortionary labor taxation in a reduced form way. There are a few additional assumptions. I assume that \( \frac{\partial \delta(B_l, \chi)}{\partial \chi} > 0 \), which implies that the more fiscally constrained the country is (the larger \( \chi \) is), the larger the deadweight loss from the bail-out is. Also I assume that \( \frac{\partial \delta(B_l, \chi)}{\partial B_l} > 0 \) which implies that the marginal cost of taxing is higher, the smaller the fiscal capacity is. The final assumption is that \( \frac{\partial \delta(B_l, \chi)}{\partial B_l \partial \chi} > 0 \). For example, a functional form that satisfies all of these conditions and will be used in the simulations is a standard convex cost function \( \delta(B_l, \chi) = \chi B_l^\eta \) where \( \eta > 1 \) and \( \chi > 0 \).

In reality, bail-outs are financed either by taxing the residents of the country during the financial crisis or by borrowing from the future. All else constant, countries with a smaller tax base or a larger sovereign debt, when the crisis state is realized, are countries that are more fiscally constrained. Larger \( \chi \) (small fiscal capacity) is a proxy for the fact that the deadweight loss from the marginal dollar taxed will be higher, the smaller the tax base is. Also larger \( \chi \) is a proxy for the fact that

\(^{16}\) There will be no role for a bail-out in this model in \( t = 2 \). Also one can relax the assumption that there are bailouts only in the crisis state. The result that larger fiscal capacity implies a lower minimum bank capital requirement remains.

\(^{17}\) This assumption captures the fact that it is the aggregate bail-out that affects the marginal cost of government borrowing or taxing.
the marginal cost of borrowing will be higher, the higher the level of sovereign debt is since the interest rate charged will be higher.\textsuperscript{18} The assumption $\frac{\partial \delta(B_1, \chi)}{\partial B_1 \partial \chi} > 0$ captures both of those cases.

The reason why the government would optimally choose to provide a bail-out to the banking sector, even though the bail-out is costly, is that a larger bail-out will reduce the size of the fire sale during financial crises. As a result, less capital will be transferred from the more productive sector (the banking sector) to the less productive sector (the consumers). The consumers end up benefiting because they receive the profits of the banks in the last period in the form of dividends. As I show later, the optimal bail-out will be determined in equilibrium by equating the marginal cost of the bail-out to the marginal benefit. The optimal bail-out will be a function of the fiscal capacity of the country. The larger the fiscal capacity is (the smaller $\chi$ is), the smaller the deadweight loss is and hence the optimal bail-out will be larger relative to a country with smaller fiscal capacity. A country with infinite fiscal capacity ($\chi = 0$) will have a zero deadweight loss from the bail-out. A country with no fiscal capacity ($\chi \to \infty$) will have an infinite deadweight loss from the bail-out. This will be a country that finds it too costly or impossible to tax and it is also completely shut off from foreign debt markets.

### 2.4 Assumptions

In addition to the assumptions made so far, I also assume that the following inequalities are satisfied. The first assumption ensures that period zero investment does not have expected return greater than one in period one if the return in the low state is zero

$$\pi_h (1 + a_{1h} - \gamma) < 1$$

(Assumption 1)

If Assumption 1 is violated, it will always be optimal to lever to the maximum in period zero and invest as much as possible which will make the problem trivial. The following assumption ensures that if there is no fire sale, the expected return on period zero investment is greater than the cost

$$1 < \sum_s \pi_s [1 - \gamma + a_{1s}]$$

(Assumption 2)

If Assumption 2 was violated, period zero investment would be zero. In order to have a fire sale in the model, it has to be the case that the fraction of capital that can be pledged, $\theta$, plus the return to period zero capital in the crisis state, $a_{1l}$, is less than the refinancing cost of capital, $\gamma$. I also assume that the refinancing cost is less than 1 which is the highest possible price of capital.

$$a_{1l} + \theta < \gamma < 1$$

(Assumption 3)

To ensure uniqueness, I also assume

$$F' (k^T_{1l}) - \theta + F'' (k^T_{1l}) k^T_{1l} > 0$$

$$F''' (k^T_{1l}) k^T_{1l} + F'' (k^T_{1l}) < 0$$

(Assumption 4)

\textsuperscript{18}In a full fledged sovereign borrowing model in the spirit of [Eaton & Gersovitz, 1981] and [Aguiar & Gopinath, 2006], the higher sovereign borrowing is, the higher the interest rate charged will be per dollar borrowed.
Assumption 5 guarantees that in the high state there is never a fire sale

\[ a_{1h} > 1 - \theta \]  
(Assumption 5)

Finally, I assume that return to period zero and period one investment is non-negative, \( A > 0, a_{1s} \geq 0 \).

### 3 Ramsey Problem Without Commitment

In this section I solve for the Ramsey Problem with no commitment. The policy maker has access to both an ex-post instrument — a bail-out during the crisis state — and an ex-ante instrument — a minimum bank capital requirement. Since bankers are infinitesimally small, they take the price of capital in \( t = 1 \), \( q_{1s} \), as given. Also they take the prices of the state contingent borrowing contracts, \( p_{1s} \) and \( p_{2s} \), as given. In equilibrium, it will never be optimal to pay dividends in \( t = 0 \) and \( t = 1 \) and, hence, I omit those choice variables from the set-up. The superscript \( i \) refers to bank \( i \). Variables without superscript are aggregates and defined as \( x = \int_0^1 x^i di \). I solve for the Ramsey Problem backwards.

The actions in reverse order are the following. In \( t = 2 \), all bankers produce and pay out all the profits as dividends to the consumers. At the end of \( t = 1 \), banker \( i \) chooses \( \{ k_{1s}^i, d_{2s}^i \} \) taking as given the state variables \( \{ B_i^i, k_{0}^i, d_{1s}^i, r^i \} \). Banker \( i \) maximizes the dividend payment in the last period

\[
\max_{k_{1s}^i, d_{2s}^i} (A + 1) k_{1s}^i - d_{2s}^i
\]

subject to the collateral constraint in \( t = 1 \)

\[
d_{2s}^i \leq \theta q_{2s} k_{1s}^i \quad \left[ \lambda_{2s}^i \right] \quad (1)
\]

where the Lagrangians are given in square brackets. Banker \( i \) also takes into account the period one budget constraint

\[
k_{1s}^i q_{1s} + d_{1s}^i \leq (q_{1s} + a_{1s} - \gamma) k_{0}^i + B_i^i + p_{2s} d_{2s}^i \quad \left[ z_{1s}^i \right] \quad (2)
\]

where \( B_i^i = 0 \) since I assumed bail-outs are prohibitively costly if there is no crisis. From the first order conditions and the constraints one can solve for \( \{ k_{1s}^i, d_{2s}^i \} \) as a function of the state variables and prices.

At the beginning of \( t = 1 \), first, banker \( i \) repays the promised debt \( d_{1s}^i \). After that, if in the low state, the policy maker chooses \( B_i^i \) given the state variables \( \{ k_{0}^i, d_{1s}^i, r^i \} \). He also takes into account that the optimal allocation chosen by the banker at the end of \( t = 1 \) is a function of \( B_i^i \) and that the equilibrium is symmetric. The objective function of the policy maker in \( t = 1 \) in the low state is
subject to the period one budget constraint in the low state given by equation 2 with a Lagrangian 
z^{1,RP}. \((A + 1) k_{1l} - d_{2l}\) are the dividends paid from the bankers to consumers in \(t = 2\) in the low state, \(F (k_{1l}^T) - F' (k_{1l}^T) k_{1l}^T + d_{1l} - p_{2l} d_{2l} + d_{2l} + (A + 1) k_{1l} - d_{2l}\),

At the end of \(t = 0\), banker \(i\) chooses \(\{k_{0i}, d_{1s}^i\}\), taking into account that the optimal bail-out in the crisis state is potentially a function of his choice variables and so is the allocation he chooses at the end of \(t = 1\). Banker \(i\) optimizes the expected value of dividends

\[
\sum_s \pi_s \left[(A + 1) k_{1s}^i - d_{2s}^i\right]
\]

subject to the budget constraints in \(t = 1\) given by equation 2, the period zero Lagrangian of which is \(\pi_s z^{i,0}\). Also banker \(i\) takes into account the period zero budget constraint

\[
k_{0i} \leq \sum_s \pi_s p_{1s} d_{1s}^i + n \quad [z^{i}_0]\]

The optimization problem is also subject to the \(t = 0\) collateral constraint

\[
d_{1s}^i \leq \theta (q_{1s} - \gamma) k_{0i} \quad [\pi_s \lambda_{1s}^i]\]

and the minimum bank capital requirement constraint

\[
\rho^i k_{0i} \leq n \quad [t^i]\]

Finally, at the beginning of \(t = 0\), the policy maker chooses \(\rho^i\) taking into account how it affects all future choice variables. In this subsection, I solve the problem assuming \(\rho^i = 0\) and later on I solve for the optimal \(\rho^i\). Details on the solution and first order conditions are provided in the Appendix, Section 10.2. Given the assumptions made on the production technology of the consumers in \(t = 2\) and the fact that the consumption good can be transformed into capital one to one in every period, it follows that \(q_{2s} = 1\). Also, in equilibrium, from the optimization problem of the consumer \(p_{2s} = p_{1s} = 1\).

**Proposition 1** Given Assumptions 1-5, considering a symmetric equilibrium for the Ramsey Problem with no commitment and assuming no ex-ante regulation \(\rho = 0\), there is never a fire sale in the high state, \(q_{1h} = 1\). Given the additional assumption

\[
(d')^{-1} \left( \frac{A}{1 - \theta} \right) < \frac{(\gamma - \theta - a_{1l} + \theta (1 - \gamma)) n}{1 - \theta (1 - \gamma)}
\]

(Assumption 6)
it is always the case that there is a fire sale in the low state, \( q_{1l} < 1 \). The equilibrium exists and is unique and is one of the following types: 1) \( z_{0}^{CE} = z_{1l}^{CE} > z_{1h}^{CE} \) (interior equilibrium) and 2) \( z_{0}^{CE} > z_{1s}^{CE} \) (corner equilibrium where the banker borrows to the maximum in \( t = 0 \)) where \( z_{1s}^{CE} = z_{1s}^{0,CE} \). The optimal bail-out is determined by

\[
1 + \delta'(B_t) = z_{1l}^{1,RP} = \frac{F''(k_{1l}^T) k_{1l}^t + A + 1 - \theta}{F'(k_{1l}^T) k_{1l}^t + F'(k_{1l}^T)} - \theta
\]  

(6)

If also Assumption 7 is satisfied (provided in the Appendix, Section 10.5.1), the only possible equilibrium is the interior equilibrium of Type 1 where

\[
z_{0}^{CE} = \sum \pi_t z_{1s}^{CE} (q_{1s} + a_{1s} - \gamma) + \lambda_{1h}^{CE} \pi_h (1 - \gamma)
\]  

(7)

\[
z_{1s}^{CE} = \frac{A + 1 - \theta}{q_{1s} - \theta}
\]  

(8)

where \( \lambda_{2s}^{CE} > 0, \lambda_{1h}^{CE} = z_{0}^{CE} - z_{1h}^{CE} > 0 \) and \( \lambda_{1l}^{CE} = 0 \).

**Proof of Proposition 1.** See Appendix, Section 10.5.1. ■

Assumption 6 guarantees that the optimal bail-out and the amount of resources that can be transferred to the crisis state using the state contingent debt contract are not too large so that there will be a fire sale in the crisis state. The key variables that characterize the equilibrium type are \( z_{1s}^{CE} \) and \( z_{0}^{CE} \). \( z_{0}^{CE} \) is the period zero marginal value of an extra unit of the consumption good (which, from now on, I will refer to as an extra dollar) as perceived by the banker. \( z_{1s}^{CE} \) is the scaled marginal value of an extra dollar in \( t = 1 \), state \( s \), as perceived by the banker. When the banker decides optimally whether to keep an extra dollar in \( t = 0 \) or to "transfer" it to \( t = 1 \) state \( s \) using a state contingent contract, the relevant variables to compare are \( z_{0}^{CE} \) and \( z_{1s}^{CE} \). If the banker "transfers" an extra dollar from \( t = 0 \) to period 1, state \( s \), he will get \( \frac{1}{\pi_s} \) units of the consumption good in \( t = 1 \), state \( s \), and his ex-ante welfare will increase by \( (\pi_s z_{1s}^{CE}) \frac{1}{\pi_s} = z_{1s}^{CE} \). If the banker keeps the dollar in \( t = 0 \), his ex-ante welfare will improve by \( z_{0}^{CE} \). Throughout the rest of the paper, I will refer to \( z_{1s}^{CE} \) and \( z_{0}^{CE} \) as the marginal value of wealth in \( t = 1 \), state \( s \), and in \( t = 0 \) respectively.\(^{19}\)

Consider the interior equilibrium (Type 1 equilibrium). It will be the case that \( z_{1s}^{CE} > 1 \) and \( z_{0}^{CE} > 1 \). This implies that in \( t = 0 \) and in \( t = 1 \), the banker optimally does not pay dividends to the consumer whose marginal value of wealth is one. Dividends will be optimally paid out only in \( t = 2 \). Also \( z_{1s}^{CE} > 1 \) implies that the banker wants to transfer the maximum amount of resources from period two to period one. As a result, the period one borrowing constraints will always bind, implying that \( \lambda_{2s}^{CE} > 0 \) and \( d_{2s} = \theta k_{1s} \). If the equilibrium is interior, then \( z_{0}^{CE} = z_{1l}^{CE} > z_{1h}^{CE} \), which implies that the banker values wealth more in period zero and in the crisis state than he values wealth in the high state in \( t = 1 \). Therefore, in period zero, the banker borrows to the maximum against the high state which implies \( \lambda_{1h}^{CE} > 0 \).

\(^{19}\)The super script \( CE \) (competitive equilibrium) implies this is the marginal value of an extra dollar in the hands of the banker, as perceived by the banker.
The first order conditions with respect to \( k_{1s} \) and \( k_0 \) determine the marginal values of wealth. Equation 8 pins down the marginal value of wealth in period one as perceived by the banker. An extra dollar in the hands of the bankers in \( t = 1 \) can purchase \( \frac{1}{q_{1s} - q} \) units of capital after levering against the capital, and the marginal benefit of an extra dollar invested is the cash flow received in \( t = 2, A \), plus the resale value of capital of one minus the debt payment \( \theta \). From the first order condition with respect to \( k_0 \), equation 7, in equilibrium, the marginal value of wealth in period zero \( z_0^{CE} \) is equated to the marginal benefit of \( k_0 \). Since the price of period zero capital is one, an extra dollar in \( t = 0 \) implies an extra unit of capital purchased in \( t = 0 \). In \( t = 1 \), state \( s \), the return from an extra dollar of period zero investment is the return \( a_{1s} \) plus the resell price, \( q_{1s} \), minus the refinancing cost, \( \gamma \). These profits get reinvested and the return is the marginal value of wealth in \( t = 1 \) state \( s \), \( z_1^{CE} \). Since the period one collateral constraint in the high state is binding, an extra \( k_0 \) has the additional benefit of relaxing the borrowing constraint which is why the marginal benefit of \( k_0 \) also includes the term \( \lambda_{1h}^{CE} \pi_b \theta (1 - \gamma) \).

In this set up, with a continuum of banks, where the bankers’ production technology is linear and the equilibrium considered is symmetric, there is no moral hazard. The first order condition of the policy maker with respect to the bail-out, equation 6, pins down the optimal \( B_l \) by equating the marginal cost of an extra dollar of bail-out \( 1 + \delta' (B_l) \) to the marginal benefit of an extra dollar in the hands of the banker in the low state in \( t = 1 \), as perceived by the policy maker, \( z_1^{RP} (k_1^{T}) \). From equation 6, it is clear that only the aggregate bail-out is pinned down, \( B_l \), and not the country specific one, \( B_l^i \). Also the marginal value of an extra dollar in the hands of the bankers in the crisis state, as perceived by the policy maker, \( z_1^{RP} \), is the same for all banks, regardless of the size of the bail-out given to each banker.\(^{20}\) In order to solve the model, I assume that the equilibrium is symmetric where the government gives the same bail-out to each bank, \( B_l = B_l^i \), and every banker internalizes that when making decisions in period \( t = 0 \). However, since \( z_1^{RP} \) is a function only of the aggregate fire sale and bankers are small, the choice of \( d_{1s}^i \) and \( k_0^i \) does not affect the marginal value of the bail-out and, hence, the bail-out received from the government. As a result, there will be no moral hazard in this particular set up, i.e. \( \frac{\partial B_l^i}{\partial k_0^i} = \frac{\partial B_l^i}{\partial d_{1s}^i} = 0 \). This result changes when the model is extended to consider only a finite number of banks in Section 6.

In order to solve for the optimal allocation, first, I solve for the optimal amount of period zero investment, \( k_0^{CE} \). I will provide intuition for the proof of existence and uniqueness using a graphical approach of how \( k_0^{CE} \) is determined. In subsequent sections, I will build on Graph 1 to prove that there is overinvestment if \( \rho = 0 \). I will also prove the key result of this paper — that smaller fiscal capacity implies optimally a larger ex-ante minimum capital ratio. Define the following function of \( k_0 \)

\[
\psi^{CE} (k_0) = z_1^{CE} (k_0) - z_0^{CE} (k_0)
\]

If the equilibrium is interior, \( k_0^{CE} \) will be pinned down by \( \psi^{CE} (k_0^{CE}) = 0 \). \( z_1^{CE} (k_0^{CE}) = z_0^{CE} (k_0^{CE}) \) implies that, in equilibrium, the banker is indifferent between transferring an extra dollar into the

\(^{20}\)This is a result of the linearity of the production technology of the bankers. Optimally, the policy maker wants to achieve an aggregate fire sale of \( k_{1i}^T = \int_0^1 k_{1i}^{T} \, dt \), which is determined by equation 6. He is indifferent whether this is achieved by each banker fireselling less capital or a few bankers purchasing capital from the rest of the bankers.
crisis state using a state contingent contract or investing an extra dollar in $t = 0$. Graph 1 depicts $\psi^{CE}(k_0)$ for $\rho = 0$.\footnote{The functional forms used for $\delta(\cdot)$ and $F(\cdot)$ are given in the beginning of the Appendix, Section 10. The parameters used to produce Graph 1 are $\gamma = 0.7$, $\alpha = 0.8$, $A = 0.8$, $a_{1h} = 1.5$, $a_{1l} = 0$, $\pi_h = .55$, $n = .5$, $\theta = .6$, $\eta = 1.5$.}

First, notice that $\psi^{CE}(k_0)$ is a strictly increasing function of $k_0$. The intuition for this result is the following. In the proof of Proposition 1 in the Appendix, I show that the larger period zero investment is, $k_0$, the larger the fire sale is, $k^T_{1l}$, and there is a one-to-one mapping between $k_0$ and $k^T_{1l}$. A larger fire sale leads to a larger marginal value of wealth in the crisis state, $z^{CE}_{1l}$, since the price of capital is lower. In addition, the marginal value of wealth in $t = 0$, $z^{CE}_0$, is lower, the larger the fire sale is since the marginal benefit of $k_0$ is lower due to the fact that the resale price of period zero capital in the low state is lower. Therefore, as $k_0$ increases, $z^{CE}_{1l}$ increases and $z^{CE}_0$ decreases, leading to $\psi^{CE}(k_0) > 0$. As a result, $\psi^{CE}(k_0)$ will cross the zero line at most once. If Assumption 7 is satisfied, the parametrization is such that the interior equilibrium is always the optimal one.

4 Constrained Central Planner’s Problem Without Commitment

In this section I solve for the constrained Central Planner’s problem without commitment. The constrained Central Planner optimizes the welfare of the consumers who are also the owners of the banks.\footnote{Alternatively, one can think of this set up as there being a representative family that splits into bankers and consumers in the beginning of period zero and the bankers are given the exogenous starting capital of $n$. The bankers and consumers reunite in $t = 2$ and consume jointly. The bankers borrow from consumers that are not members of their family. The latter interpretation is in the spirit of [Gertler & Kiyotaki, 2010].} The most important difference between the Central Planner’s problem and the banker’s problem is that the Central Planner takes into account the impact of her decisions on the price of capital, which is the source of the pecuniary externality. Also the Central Planner internalizes...
both the cost and the benefit of the bail-out, unlike the banker in the competitive equilibrium who internalizes only the benefit. Finally, the Central Planner takes into account that the firesold capital improves the welfare of the consumer via the profits from operating the consumer’s production technology given by \( F \left( k_{1s}^T \right) - F' \left( k_{1s}^T \right) k_{1s}^T \). The Central Planner faces the same constraints as the banker.

I solve the problem backwards taking into account that \( p_{1s} = p_{2s} = 1 \), \( q_{1s} = F' \left( k_{1l}^T \right) \) and \( q_{2s} = 1 \). In period one, the Central Planner chooses \( \{B_l, k_{1s}, d_{2s}\} \) as a function of the state variables \( \{k_0, d_{1s}\} \). He maximizes the following objective function

\[
2e + F \left( k_{1s}^T \right) - F' \left( k_{1s}^T \right) k_{1s}^T + d_{1s} - (B_s + \delta (B_s)) + (A + 1) k_{1s} - d_{2s}
\]

subject to the collateral constraint in period one, equation 1, with a Lagrangian given by \( \lambda_{2s}^{CP} \) and the period one budget constraint, equation 2, with a Lagrangian given by \( z_{1s}^{CP} \).

In \( t = 0 \), the Central Planner chooses \( \{k_0, d_{1s}\} \) in order to optimize the following ex-ante welfare function

\[
3e + \sum \pi_s \left[ F \left( k_{1s}^T \right) - F' \left( k_{1s}^T \right) k_{1s}^T - (B_s + \delta (B_s)) + (A + 1) k_{1s} - d_{2s} \right]
\]

The period zero optimization problem is subject to the budget constraint in \( t = 1 \), equation 2, with a Lagrangian given by \( z_{1s}^{0,CP} \) and the budget constraint in \( t = 0 \), equation 3, where the Lagrangian is \( z_0^{CP} \). The Central Planner also takes into account the period zero collateral constraint given by equation 4 and the Lagrangian is \( \pi_s \lambda_{1s}^{CP} \). For details on the set up and the solution see Appendix, Section 10.3.

**Proposition 2** Given Assumptions 1-6, considering a symmetric equilibrium for the Central Planner’s problem with no commitment, there is never a fire sale in the high state, \( q_{1h} < 1 \). The equilibrium exists and is unique and is one of the following two types. 1) \( z_1^{CP} = z_{1t}^{CP} > z_{1h}^{CP} \) (interior equilibrium) and 2) \( z_0^{CP} > z_1^{CP} \) (corner equilibrium where the banker borrows to the maximum in \( t = 0 \)) where \( z_1^{CP} = z_0^{1,CP} \). The optimal bail-out is determined by

\[
1 + \delta' (B_l) = z_{1l}^{1,CP} = \frac{F'' \left( k_{1l}^T \right) k_{1l}^T + A + 1 - \theta}{F'' \left( k_{1l}^T \right) k_{1l}^T + F' \left( k_{1l}^T \right) - \theta}
\]  

(9)

If also Assumption 7’ is satisfied (provided in the Appendix, Section 10.5.2), the only possible equilibrium is the interior equilibrium of Type 1 where

\[
\sum \pi_s \left( -F'' \left( k_{1s}^T \right) k_{1s}^T + z_{1s}^{CP} \left( F' \left( k_{1s}^T \right) + \alpha_{1s} - \gamma + F'' \left( k_{1s}^T \right) k_{1s}^T \right) \right) + \pi_h \lambda_{1h}^{CP} \theta [1 - \gamma] = z_0^{CP}
\]  

(10)

\[
z_{1h}^{CP} = \frac{A + 1 - \theta}{1 - \theta}
\]  

(11)

\[
z_{1l}^{CP} = \frac{F'' \left( k_{1l}^T \right) k_{1l}^T + A + 1 - \theta}{F' \left( k_{1l}^T \right) - \theta + F'' \left( k_{1l}^T \right) k_{1l}^T}
\]  

(12)
\[ \lambda_{s}^{CP} > 0, \lambda_{1h}^{CP} = z_{0}^{CP} - z_{1h}^{CP} > 0 \text{ and } \lambda_{1h}^{CP} = 0. \]

**Proof of Proposition 2.** See Appendix, Section 10.5.2. ■

The first order conditions with respect to \( k_{0} \) and \( k_{1s} \) determine the marginal value of wealth in the hands of the banker as perceived by the Central Planner, \( z_{0}^{CP} \) and \( z_{1s}^{CP} \). First, consider the first order condition with respect to \( k_{1l} \) — equation 12. An extra dollar in the hands of the bankers in \( t = 1 \) in the crisis state can purchase \( \frac{1}{q_{1s} - \theta} \) units of capital after levering against the capital. Also the Central Planner realizes that his actions affect the price at which the capital is purchased and hence the \( F''(k_{1l}^{T})k_{1l}^{T} \) term in the denominator. In other words, the Central Planner realizes that larger \( k_{1l} \) relaxes the budget constraint of all bankers by increasing the price of capital. The marginal benefit of an extra dollar invested in \( t = 1 \) is the period two return, \( A \), plus the resale value of capital of one minus the debt payment \( \theta \). Larger \( k_{1l} \) also implies smaller resale and lower profits for the consumer, hence, the \( F''(k_{1l}^{T})k_{1l}^{T} \) term in the numerator.

From the first order condition with respect to \( k_{0} \), equation 10, in equilibrium, the marginal value of wealth in period zero \( z_{0}^{CP} \) is equal to the marginal benefit of \( k_{0} \), as in the Ramsey Problem. In \( t = 1 \) and state \( s \), the marginal profit of \( k_{0} \) is \( q_{1s} + a_{1s} - \gamma + F''(k_{1s}^{T})k_{1s}^{T} \) which consists of the return, \( a_{1s} \), plus the resale price minus the refinancing cost. The additional term, \( F''(k_{1s}^{T})k_{1s}^{T} \), enters the marginal profit of \( k_{0} \) because the Central Planner internalizes the fact that an extra \( k_{0} \) implies larger resale, tighter budget constraints for all bankers and, hence, lower prices. The period one profits are reinvested and the return is the marginal value of wealth in \( t = 1 \) state \( s \) as perceived by the Central Planner, \( z_{1s}^{CP} \). The term, \( -F''(k_{1s}^{T})k_{1s}^{T} > 0 \), captures the fact that an additional \( k_{0} \) will lead to a larger resale and higher profits for the consumers. Since the period zero collateral constraint in the high state is binding, an extra \( k_{0} \) has the additional benefit of relaxing the borrowing constraint, which is why the marginal benefit of \( k_{0} \) also includes the term \( \lambda_{1h}^{CE} \pi_{h}\theta (1 - \gamma) \).

There are a few key results one should note from the constrained Central Planner’s solution. The first result is that the constrained Central Planner’s problem has the same two types of equilibria as the competitive equilibrium where the interior equilibrium, as before, will be the one of interest. Also both the constrained Central Planner and the banker optimally choose the same pecking order of borrowing in the sense that, in period zero, they borrow to the maximum against the high state and only then they might borrow against the low state. This follows from the fact that both the banker and the Central Planner value an extra dollar in the low state more than they value an extra dollar in the high state. This result is what allows us to de-centralize the constrained Central Planner’s allocation using only a single ex-ante instrument — a minimum bank capital requirement.

The second key result is that the Central Planner values an extra dollar in the hands of the banker in the crisis state by more than the banker in the Ramsey Problem does. Assuming an interior equilibrium for the Central Planner (Assumption 7’ is satisfied), since \( q_{1l} < 1 \), it follows that

\[ z_{1l}^{CP} > z_{1l}^{CE} \quad (13) \]
The intuition behind this result is that the constrained Central Planner internalizes the fact that an extra dollar in the hands of the bankers in the crisis state will lead to a smaller fire sale. This will improve the balance sheet of all other bankers by relaxing their budget constraints, leading to an even lower aggregate fire sale. As a result, less capital will be transferred from the more productive users of capital — bankers — to the less productive users of capital — consumers — which improves aggregate welfare. As I show in the next subsection, this channel is at the heart of the pecuniary externality and the overinvestment.

5 Compare the Central Planner’s and the Competitive Equilibrium Allocation

In this section, first, I compare the constrained Central Planner’s allocation without commitment and the allocation from the Ramsey Problem without commitment if \( \rho = 0 \). I prove that there is overinvestment due to pecuniary externalities. Second, I show that if the policy maker has an access to a minimum bank capital requirement, the competitive equilibrium from the Ramsey Problem coincides with the constrained Central Planner’s allocation.

**Proposition 3** Conditional on assumptions 1-6 and comparing the constrained Central Planner’s allocation without commitment and the Competitive Equilibrium from the Ramsey Problem without commitment where \( \rho = 0 \), there is always overinvestment, \( k_0^{CP} < k_0^{CE} \), if the equilibrium is of Type 1 for the Central Planner (interior equilibrium) and there is no overinvestment, \( k_0^{CP} = k_0^{CE} \), if the equilibrium is of Type 2 for the Central Planner (corner equilibrium).

**Proof of Proposition 3.** See Appendix, Section 10.5.3. ■

The graphical representation of the proof is the following. Assume that the equilibrium is of Type 1 for the Central Planner and define \( \psi^{CP}(k_0) = z_1^{CP}(k_0) - z_0^{CP}(k_0) \). Graph 2 depicts \( \psi^{CE}(k_0) \) and \( \psi^{CP}(k_0) \), using the same parametrization as in Graph 1.
Similarly to \( \psi^{CEl}(k_0) > 0 \), one can prove that \( \psi^{CPi}(k_0) > 0 \). As before, larger \( k_0 \) implies larger fire sale, \( k_T^l \) and the price of capital is lower. As a result, more wealth is transferred from the more productive to the less productive sector and, hence, the marginal value of wealth in the crisis state as perceived by the Central Planner, \( z_{lT}^{CP} \), is larger. At the same time, \( z_0^{CP} \), which in equilibrium equals the marginal benefit of an extra \( k_0 \) as perceived by the Central Planner, decreases with the fire sale. The reason is precisely the same as to why \( z_{lT}^{CP} \) increases with the fire sale (For details on the derivations see the proof of Proposition 3).

In order to prove that there is overinvestment, it is sufficient to show that as long as the equilibria of both the Central Planner’s problem and the Ramsey problem are of Type 1, \( \psi^{CP}(k_0) > \psi^{CE}(k_0) \). The reason why this is true is that, for a given \( k_0 \), the constrained Central Planner values wealth in the hands of the banker in the crisis state by more than the banker does, \( z_{lT}^{CP} > z_{lT}^{CE} \), which is shown by equation 13. In addition, \( z_0^{CP} < z_0^{CE} \) because the Central Planner, unlike the individual banker, internalizes the fact that the larger \( k_0 \) is, the larger the fire sale will be, and the larger the transfer of capital will be from the more productive to the less productive sector. As a result, the perceived marginal benefit of period zero investment is lower for the Central Planner than for the banker.\(^{2425}\)

Proposition 4 makes the point that the constrained Central Planner’s allocation can be decentralized using a minimum capital requirement.

**Proposition 4** Conditional on assumptions 1-6 and an interior equilibrium for the Central Planner (Type 1 equilibrium), the constrained Central Planner’s allocation can be decentralized using

\(^{23}\)Notice that if the equilibrium of the Central Planner’s problem is of Type 1 and the equilibrium of the Ramsey problem is of Type 2, then it is obvious that there is overinvestment since Type 2 equilibrium occurs when period zero investment is the maximum amount possible.

\(^{24}\)It is a well known fact that in a standard Arrow Debreu economy with no frictions, where agents are small and take prices as given, there are no pecuniary externalities. The reason is that, in that model, the change in the price is just a wealth transfer from one agent to another and, in equilibrium, the marginal utility of wealth across agents is equalized, implying that the net effect on welfare is zero. This is why the assumption that bankers are more productive than consumers (i.e. they have different marginal valuations of wealth) is crucial for the pecuniary externalities.

\(^{25}\)One can also use a perturbation argument to show why conditional on \( \rho = 0 \), if \( k_0^{CE} \) is decreased marginally, ex-ante welfare will improve. The direct effect of decreasing initial investment is zero since \( z_0^{CE} = z_{lT}^{CE} \) and the banker is indifferent whether he saves or invests the extra dollar. However, the indirect effect of decreasing initial investment is positive. Lower ex-ante investment increases the price of capital by \( dq_{lT} = -F''(k_T^l) > 0 \). The banker’s ex-ante value of profits/dividends paid increases by \( \pi z_{lT}^{CE} k_T^l dq_{lT} \) while consumers’ ex-ante value of profits from running the outside technology decreases by \( \pi_T k_T^l dq_{lT} \). Since consumers own the banks and receive dividends in \( t = 2 \) and in \( t = 2 \) the marginal value of wealth in the hands of the bankers is one, ex-ante welfare improves by \( \pi_T (z_{lT}^{CE} - 1) k_T^l dq_{lT} > 0 \) since \( z_{lT}^{CE} > 1 \). Notice that the term \( \pi_T (z_{lT} - 1) k_T^l dq_{lT} \) is what differentiates the first order condition of the banker with respect to \( k_0 \) from the first order condition of the Central Planner with respect to \( k_0 \).

In [Lorenzoni, 2008] the interior equilibrium where \( z_0^{CE} = z_{lT}^{CE} \) exhibits no overinvestment since the transfers are ex-ante rather than ex-post. Since in [Lorenzoni, 2008] the consumers have to be compensated in \( t = 0 \) for receiving lower profits in the crisis state, the second order effect is zero and there is no ex-ante welfare Pareto improving deviation.
only one ex-ante instrument — a minimum capital requirement where the optimal minimum capital requirement is given by \( \rho^* = \frac{n}{k_0^{CP}} \).

**Proof of Proposition 4.** See Appendix, Section 10.5.4. ■

Since there is always overinvestment if the equilibrium is interior for the Central Planner, the minimum capital requirement constraint will be always binding. As a result, if the Central Planner sets \( \rho^* = \frac{n}{k_0^{CP}} \), the banker will optimally choose to invest \( k^{CE}_0 = \frac{n}{\rho^*} = k_0^{CP} \). In that sense, the minimum capital ratio is a "quantity" regulatory instrument since it directly determines the quantity of period zero investment chosen by the banker. The banker has another degree of freedom, which is to choose how to transfer resources across states of nature and time. More precisely, he can choose how much liquidity to transfer to the crisis state. The amount of period zero borrowing is given by \( \sum \pi_s d_{1s} = k_0^{CP} - n \) and also \( d_{2s} \) is determined by the borrowing constraint in \( t = 2 \) which is binding. Hence, what is left to decide is the state contingent contract in period zero, \( \{d_{1l}, d_{1h}\} \), that satisfies \( \sum \pi_s d_{1s} = k_0^{CP} - n \). Both the Central Planner and the banker have the same pecking order of borrowing. They both value wealth more in the low state in \( t = 1 \) than in the high state in \( t = 1 \). As a result, they both choose to optimally borrow to the maximum first against the high state, \( d_{1h} = \theta (1 - \gamma) k_0 \), and only then against the low state. Therefore, a second instrument is not required in order to decentralize the constrained Central Planner’s allocation. I show that this result changes when moral hazard is introduced in the model and the moral hazard is strong enough.

### 5.1 Relationship Between Fiscal Capacity and Optimal \( \rho^* \)

In this sub-section, I prove the key result of this paper — that when the source of the ex-ante inefficiency and overborrowing is pecuniary externalities and the ex-ante instrument available is a minimum bank capital requirement, smaller fiscal capacity implies a larger optimal minimum capital requirement.

**Proposition 5** Conditional on Assumptions 1-6 and on an interior equilibrium for the Central Planner (Assumption 7’), smaller fiscal capacity (larger \( \chi \)) optimally implies a higher bank capital ratio \( \rho^* \), \( \frac{\partial \rho^*}{\partial \chi} > 0 \).

**Proof of Proposition 5.** See Appendix, Section 10.5.5. ■

I already proved in Proposition 4 that the constrained Central Planner’s allocation can be decentralized using a single ex-ante instrument and that the optimal bank capital ratio, \( \rho^* \), and \( k_0^{CP} \) are inversely related. As a result, it is sufficient to prove that the Central Planner of a country with a larger fiscal capacity will choose to optimally invest more ex-ante relative to a country with a smaller fiscal capacity. I present the intuition of why this is the case graphically.\(^{26}\)

\(^{26}\)The rest of the parameters are the same as in Graph 1.
Larger Fiscal Capacity Implies Larger Ex-Ante Investment

\[ \psi_{CP} = z_{CP}^1 - z_{CP}^0; \chi = 10 \]

\[ \psi_{CP} = z_{CP}^1 - z_{CP}^0; \chi = 20 \]

Graph 3 plots \( \psi_{CP}(k_0, \chi) \) for two countries with different fiscal capacity. Larger \( \chi \) implies smaller fiscal capacity. Given that \( \psi_{CP}(k_0) > 0 \) it is sufficient to prove that, for a given \( k_0 \), \( \frac{\partial \psi_{CP}(\chi k_0)}{\partial \chi} > 0 \) (partial derivative). The reason why this is true is that the smaller the fiscal capacity is (\( \chi \) is larger), for a given \( k_0 \), the smaller the ex-post bail-out is, and the larger the fire sale is in a crisis. As a result, \( z_{CP}^1 \) is larger for the country with the smaller fiscal capacity. Furthermore, the marginal benefit of period zero investment, which in equilibrium is equal to \( z_{CP}^0 \), is also smaller for the country with the smaller fiscal capacity, for a given \( k_0 \), since the larger the fire sale is, the more capital is transferred from the more productive to the less productive sector. This is the intuition for the proof of why a country with a larger fiscal capacity can optimally afford to have larger investment booms ex-ante, given that it can intervene more ex-post and prop up prices. As long as the constrained Central Planner’s allocation is replicated, in order to determine the optimal \( \rho^* \), the first order condition of the banker from the Ramsey Problem with respect to \( k_0 \) will not be relevant.\(^{27}\)

Graph 4 presents the key result visually for the same set of parameters as in Graph 1. I have imposed Assumption 7’ so that the equilibrium is of Type 1 for the Central Planner and, as a result, there is overinvestment.

\(^{27}\)The first order condition of the banker from the Ramsey Problem will be used only to pin down the Lagrangian on the minimum bank capital requirement constraint, \( \ell \).
6 Introducing Moral Hazard – A Finite Number of Banks

So far, I assumed that there is a continuum of bankers. However, in reality, banking sectors are fairly concentrated. In this section, I assume that there is a finite number of banks, \( N \), instead of a continuum of banks. Since bankers are large, they internalize partially their impact on the aggregate fire sale, and, as a result, on the bail-out received. Therefore, the assumption of a finite number of banks introduces moral hazard into the model. Also the fewer the banks are, the smaller the pecuniary externality is, since bankers realize they are affecting the price of "firesold" capital.

In this section, I will show that the moral hazard is stronger for economies with more concentrated banking systems (a few banks) and a larger fiscal capacity. Also I will prove that a single ex-ante instrument — a minimum capital requirement — will be no longer sufficient to replicate the constrained Central Planner’s allocation for countries that have strong moral hazard. Those countries will be tempted to transfer too little liquidity into the crisis state, in order to increase the size of the fire sale and, hence, the bail-out received. As a result, in order to achieve the constrained Central Planner’s allocation, the policy maker will have to impose a second ex-ante instrument in the form of a minimum liquidity requirement. Finally, I will show that the key result of this paper — that smaller fiscal capacity implies optimally a larger minimum bank capital requirement still holds in the presence of moral hazard.

I solve for the Ramsey Problem with \( N \) banks, which is similar to the case with a continuum of banks. The only difference is that now bankers partially internalize their impact on the firesold capital and, hence, on the price of capital during crises. The smaller \( N \) is, the larger the individual impact is. All aggregate variables are defined as averages; \( k_{lT} = \sum_{i=1}^{N} \frac{1}{N} k_{iT} \) and \( B_{lT} = \sum_{i=1}^{N} \frac{1}{N} B_{li} \). The Ramsey Problem with \( N \) banks coincides with the Ramsey Problem with a continuum of banks as long as \( N \to \infty \). Details of the exact set up and the solution of the Ramsey Problem are provided.
in Section 10.4 of the Appendix. Proposition 6 specifies the conditions under which the equilibrium exists and is unique.

**Proposition 6** Conditional on Assumptions 1-6, no ex-ante regulation, considering a symmetric equilibrium for the Ramsey Problem with $N$ banks and no commitment, there is never a fire sale in the high state, $q_{1h} = 1$, and there is a fire sale in the low state, $q_{1l} < 1$. Conditional on the assumptions that $F''(k_{1l}^T) = 0$ and

$$(1 + \frac{1}{N}) F''(k_{1l}^T) - \left[ \frac{\delta''(B_l) \left( 1 - 2z_{1l}^{1,RP} \right)}{\left[ \delta''(B_l) \right]^2} + 4 \right] \frac{\left[ F''(k_{1l}^T) \right]^2 \left( 1 - 2z_{1l}^{1,RP} \right)}{\left( F''(k_{1l}^T) k_{1l}^T + F'(k_{1l}^T) \right) - \theta^2 \delta''(B_l) N} < 0$$

(Assumption 8)

where $z_{1l}^{1,RP} = \frac{F''(k_{1l}^T) k_{1l}^T + A + 1 - \theta}{F''(k_{1l}^T) k_{1l}^T + F'(k_{1l}^T) - \theta}$ and $B_l = (\delta')^{-1} \left( z_{1l}^{1,RP} - 1 \right)$, the equilibrium is unique and exists.

**Proof of Proposition 6.** See Appendix, Section 10.5.6. ■

Assumption 8 guarantees that it is still the case that the larger the fire sale is, the larger the marginal value of an extra dollar in the crisis state is, as perceived by the banker, $\frac{\partial z_{1l}^{1,RP}}{\partial k_{1l}^T} > 0$. The assumption $F''(k_{1l}^T) = 0$ is mostly for simplification. The optimal aggregate bail-out is pinned down by the same first order condition as in the case with a continuum of banks. Imposing a symmetric equilibrium, $B_i^j = B_l$ and $\delta' (B_i^j, \chi) = z_{1l}^{1,RP} (k_{1l}^T) - 1$. When there are only a few banks, the banker partially internalizes the fact that the larger his individual fire sale is, the larger the optimal bail-out is

$$\frac{\partial B_i^j}{\partial k_{1l}^T} = \frac{1}{\delta''(B_i^j, \chi) N} \frac{\partial z_{1l}^{1,RP}}{\partial k_{1l}^T} (k_{1l}^T) > 0$$

$\frac{\partial B_i^j}{\partial k_{1l}^T} > 0$ follows from the fact that if the fire sale is large, the policy maker values an extra dollar in the hands of the banker in a crisis by more, $\frac{\partial z_{1l}^{1,RP} (k_{1l}^T)}{\partial k_{1l}^T} > 0$, which implies that the fire sale increases the expected marginal benefit from the bail-out. However, for countries with a large number of banks, the impact of the individual fire sale on the aggregate fire sale is smaller, which is captured by the $\frac{1}{N}$ term. Finally, I assumed that the marginal cost of the bail-out increases with the size of the bail-out, $\delta'' (B_i^j, \chi) > 0$. Therefore, in order for the marginal benefit of the bail-out to be equated to the marginal cost, in equilibrium, larger fire sale will imply larger bail-out.

There are two ways in which the banker can generate a larger fire sale, which will increase the size of the expected bail-out if a crisis occurs. On the one hand, banker $i$ can invest more in period zero which will lead to a larger fire sale, $\frac{\partial k_{i,0}^T}{\partial k_{0}^T} > 0$. On the other hand, he can promise a larger payment to the consumers in the the crisis state (transfer too little liquidity to the crisis state) which will decrease his net worth during a crisis and will also lead to a larger fire sale, $\frac{\partial k_{i,1}^T}{\partial \theta_{1l}^T} > 0$.  

\[28\] In the simulations, I use a functional form for $F (\cdot)$ where $F'''(k_{1l}^T) > 0$ but $F''(k_{1l}^T)$ is fairly small.
Corollary 1 If $1 < N < \infty$ and $\chi < \infty$, there is moral hazard, i.e. $\frac{\partial B^i}{\partial k_0} = \frac{\partial B^i}{\partial k^T_{11}} > 0$ and $\frac{\partial B^i}{\partial k^T_{1i}} = \frac{\partial B^i}{\partial k^T_{11}} > 0$. Also for a given $k^T_{11}$, the fewer the banks are and the larger the fiscal capacity is, the larger the moral hazard is; $\frac{\partial B^i}{\partial k_0} > 0$, $\frac{\partial B^i}{\partial k^T_{11}} > 0$ and $\frac{\partial B^i}{\partial k^T_{1i}} < 0$.

Proof of Corollary 1. See Appendix, Section 10.5.7.

Corollary 1 states that as long as the country has some fiscal capacity $\chi < \infty$ and the number of banks is finite and greater than one, the banker internalizes the fact that his period zero actions affect the size of the bail-out, which implies that there is moral hazard. The moral hazard is captured by $\frac{\partial B^i}{\partial k_0} > 0$ and $\frac{\partial B^i}{\partial k^T_{11}} > 0$. A large fiscal capacity and a few banks exacerbate the moral hazard problem. The intuition is that when the banks are large (small number of banks), they know that their marginal impact on the fire sale is large and, as a result, they are affecting the optimal bail-out by more. Similarly, if a country has a large fiscal capacity (small $\chi$), it can afford to provide a larger bail-out for a given level of the fire sale, which implies that the moral hazard is stronger.

Next I prove that in the case of a finite number of banks, the banker will overinvest relative to the constrained Central Planner if the Central Planner’s equilibrium is of Type 1 due to two reasons – moral hazard and pecuniary externalities.

Proposition 7 Conditional on Assumptions 1-6 and 8 and also assuming $A > 1 - \theta$ and $N > 1$, comparing the constrained Central Planner’s allocation without commitment and the Competitive Equilibrium from the Ramsey Problem with $N$ banks, no commitment, no minimum capital requirement and no minimum liquidity requirement, there is always overinvestment, $k^C_P < k^C_{E,N}$, if the equilibrium is of Type 1 for the Central Planner (interior equilibrium) and there is no overinvestment, $k^C_P = k^C_{E,N}$, if the equilibrium is of Type 2 for the Central Planner (corner equilibrium).

Proof of Proposition 7. See Appendix, Section 10.5.7.

The assumptions $A > 1 - \theta$ and $N > 1$ guarantee that even if there was no moral hazard, there will still be overinvestment due to the pecuniary externalities. In the case with moral hazard, the marginal value of wealth in the crisis state, as perceived by the banker, is given by

$$z^C_{E,N} = \frac{A + 1 - \theta}{N} \tilde{F}''(k^T_{11}) k^T_{11} - \frac{\partial B^i}{\partial k^T_{1i}} + \tilde{F}''(k^T_{11}) - \theta$$

(14)

The interpretation of equation 14 is that an extra dollar in the hands of the banker in the crisis state will be spent to purchase one extra unit of $k_{11}$ at the price of $\tilde{F}''(k^T_{11})$. Given that the banker is no longer small, he internalizes the fact that he affects the price which is captured by the term $\frac{1}{N} \tilde{F}''(k^T_{11}) k^T_{11}$. However, an extra dollar in the crisis state also decreases the bail-out by $\frac{\partial B^i}{\partial k^T_{1i}} = -\frac{1}{N} \tilde{F}''(k^T_{11}) k^T_{11} < 0$ since $k^T_{11}$ is larger and the fire sale is smaller. Finally the banker will
lever against the extra dollar invested in the crisis state, which is captured by the $-\theta$ term in the denominator. An extra unit of $k_{1t}$ will produce $A$ in $t = 2$ and it can be resold at the price of 1, where the $-\theta$ term in the numerator captures the fact that the banker has to repay the period one loan. If $N \to \infty$, equation 14 coincides with the case with a continuum of banks, given by equation 8. The intuition for why there is overinvestment is similar to the case with a continuum of banks. It relies on the fact that the Central Planner values an extra dollar in the hands of the banker in the crisis state by more than the banker values an extra dollar in the crisis state $z_{1t}^{CE,N} < z_{1t}^{CP}$. However, now this will be the case not only because of the pecuniary externalities, but also due to the moral hazard captured by the term $\partial B_i / \partial k_{1t}$.

The question remains whether only a minimum capital requirement will be sufficient to decentralize the constrained Central Planner’s allocation given the presence of moral hazard. Proposition 8 shows that for countries with strong moral hazard (small $N$ and small $\chi$), a minimum capital requirement will not be sufficient to achieve the constrained Central Planner’s allocation. For those countries, a minimum liquidity requirement will be also required.

**Proposition 8** Given Assumptions 1-6.8, for a given exogenous $\hat{\rho} > \frac{n}{k_{0}^{CE,N}}$, assuming no ex-ante minimum capital requirement and considering a symmetric competitive equilibrium, the equilibrium from the Ramsey Problem with $N$ banks can be one of the following four types:

1) Type 1: $z_{1t}^{CE,N} = z_{0}^{CE,N} > z_{1h}$ if $k_{1t}^{T} \in [\hat{k}_{1t}^{T}, \hat{k}_{1t}^{T, max})$

2) Type 2: $z_{0}^{CE,N} > z_{1h}$ if $k_{1t}^{T} = \hat{k}_{1t}^{T, max}$

3) Type 3: $z_{0}^{CE,N} = z_{1h} = z_{1t}^{CE,N}$ if $k_{1t}^{T} = \hat{k}_{1t}^{T}$

4) Type 4: $z_{1h} = z_{0}^{CE,N} > z_{1t}^{CE,N}$ if $k_{1t}^{T} \in [0, \hat{k}_{1t}^{T, max})$ where $\hat{k}_{1t}^{T, max}$ is pinned down by equation 33 in the Appendix. $\hat{k}_{1t}^{T}$ is unique and exists and if $0 < \hat{k}_{1t}^{T} < \hat{k}_{1t}^{T, max}$, $\hat{k}_{1t}^{T}$ is pinned down by $M(\hat{k}_{1t}^{T}) = 0$ where

$$M(\hat{k}_{1t}^{T}) = \frac{1}{N} \hat{F}''(\hat{k}_{1t}^{T}) k_{1t}^{T} + \frac{1}{\delta''(B_1)} N \frac{\partial z_{1h}^{1, RP}}{\partial k_{1t}^{T}} + \hat{F}'(\hat{k}_{1t}^{T}) - 1$$

$k_{0}^{CE,N}$ is the optimal period zero investment chosen by the banker if there is no minimum capital requirement and no minimum liquidity requirement.

**Proof of Proposition 8.** See Appendix, Section 10.5.8. ■

In Proposition 7, I showed that the banker would optimally want to invest more than the constrained Central Planner in $t = 0$, while Proposition 8 states that the banker might also be tempted to transfer too little liquidity to the crisis state relative to the constrained Central Planner.

The intuition for the latter is the following. The condition that the exogenous $\hat{\rho}$ is such that $\hat{\rho} > \frac{n}{k_{0}^{CE,N}}$ guarantees that the minimum capital requirement constraint will be always binding. The equilibrium can be one of four types. What differentiates the different types is how much the banker values wealth in the high state in $t = 1$ relative to the crisis state (the low state in $t = 1$).
In Proposition 2, I already proved that the Central Planner always values wealth more in the high state than in the crisis state. As a result, the only two possible borrowing contracts for the Central Planner are of type 1 and 2. In both of them, the Central Planner always borrows first to the maximum against the high state and only then borrows against the low state. The reason why the banker would deviate from the optimal borrowing contract that the Central Planner would choose is that he realizes that his return will be higher, the lower his net worth is in the crisis state since it leads to a larger fire sale and a larger bail-out. As Corollary 1 showed, larger $d_{Hl}^i$ will contribute to a lower net worth and a larger fire sale and, hence, a larger bail-out. This explains why the banker might value wealth less (or equally) in the crisis state relative to the high state which is why the equilibrium could be of Types 3 and 4. For example, in the equilibrium of Type 4, the banker borrows first to the maximum against the low state, $d_{Hl}^i = \theta(q_{Hl} - \gamma)k_{0}^{i}$, and only then borrows against the high state.

What determines which type of borrowing contract is the optimal one for the banker is whether the optimal $k_{1l}^{T}$ is such that $z_{1l}^{C.E,N} \leq k_{1h}^{C.E,N}$. This problem maps into determining whether the optimal $k_{1l}^{T}$ is such that $M(k_{1l}^{T}) \overset{\geq}{\leq} 0$. Notice that given Assumption 8, one can prove that $M(k_{1l}^{T})$ is strictly decreasing, $\frac{\partial M(k_{1l}^{T})}{\partial k_{1l}^{T}} < 0$. Let’s consider an interior $\hat{k}_{1l}^{T}$, i.e. $0 < \hat{k}_{1l}^{T} < \hat{k}_{1l}^{T, max}$ which is pinned down by $M(\hat{k}_{1l}^{T}) = 0$. If the optimal $k_{1l}^{T*}$ is such that $M(k_{1l}^{T*}) > 0$, the equilibrium is of Type 4 and $k_{1l}^{T*} \in (0, \hat{k}_{1l}^{T})$. If the optimal $k_{1l}^{T*}$ is such that $M(k_{1l}^{T*}) = 0$, the equilibrium is of Type 3 and $k_{1l}^{T*} = \hat{k}_{1l}^{T}$. If the optimal $k_{1l}^{T*} < \hat{k}_{1l}^{T, max}$ is such that $M(k_{1l}^{T*}) < 0$, the equilibrium is of Type 1 and $k_{1l}^{T*} \in (\hat{k}_{1l}^{T}, \hat{k}_{1l}^{T, max})$. As a result, the banker’s optimal borrowing contract might differ from the Central Planner’s, if there is a strong moral hazard and a second instrument will be required to replicate the constrained Central Planner’s allocation.

As $N$ increases or as $\chi$ increases (as the moral hazard decreases), the $M(k_{1l}^{T})$ function shifts to the left leading to the $[0, \hat{k}_{1l}^{T}]$ region becoming smaller, which makes it less likely for the optimal $k_{1l}^{T*}$ to be in the region where the banker values wealth in the crisis state less than he values wealth in the high state in $t = 1$. If $\chi \rightarrow \infty$, which implies no bail-out and no moral hazard, then $M(k_{1l}^{T}) < 0$ for every $k_{1l}^{T}$ implying that $\hat{k}_{1l}^{T} = 0$ and the only two possible types of equilibria are of Types 1 and 2, similarly to the Central Planner’s problem. Another comparative static that shuts off the moral hazard channel is $N \rightarrow \infty$. If $N \rightarrow \infty$, $M(0) = 0$ implying that $\hat{k}_{1l}^{T} = 0$, which is why with a continuum of banks only the interior equilibrium of Type 1 and the corner equilibrium of Type 2 can occur.

Next I consider parametrization where the optimal allocation is of Type 1 for the Central Planner (Assumption 7’ is satisfied). Given that, in Proposition 7 I proved that the banker always overinvests relative to the Central Planner. This implies that the minimum capital requirement is always binding, and in order to replicate the optimal period zero investment of the Central Planner, the policy maker can simply set $\hat{\rho}^{CP} = \frac{n}{k_{0}^{C.P}} > \frac{n}{k_{0}^{CP}}$. Once again, notice that the minimum capital requirement is a "quantity" instrument and by setting $\rho$, the policy maker is choosing $k_{0}^{C.E}$.

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30 See the proof of Proposition 6 in the Appendix, Section 10.5.6.
However, for countries with larger moral hazard, the constrained Central Planner’s allocation will not be replicated despite $k_{CE}^0 = k_{CP}^0$, because these countries will optimally transfer too little liquidity into the crisis state, i.e. $-d_{IL}^{CE} < -d_{IL}^{CP}$. This is what is depicted in Graph 5.³¹

When plotting Graph 5, I impose the condition $\rho^{CP} = \frac{n}{k_0^{CP}}$, which implies that $k_{CE}^0 = k_{CP}^0$. Notice that given that $N = 3$, for small values of $\chi$, (strong moral hazard), the banker optimally chooses to transfer too little liquidity into the low state relative to the Central Planner, $-d_{IL}^{CE} < -d_{IL}^{CP}$, because he realizes that lower net worth will lead to a larger bail-out in the crisis state. If the dashed line is above the solid line, the borrowing contract for the banker is either of Type 4 or Type 3 while for the Central Planner the borrowing contract is always of Type 1.

In summary, the moral hazard can work in two different dimensions. On the one hand, the banker is tempted to invest too much in $t = 0$ relative to the Central Planner and, on the other hand, the banker might be also tempted to transfer too little liquidity into the crisis state if the moral hazard is strong. As a result, a second ex-ante instrument in the form of a minimum liquidity requirement, in addition to the minimum capital requirement, is necessary to replicate the constrained Central Planner’s allocation for countries with strong moral hazard (countries with large fiscal capacity and a small number of banks).

Assume that the policy maker has an access to both a minimum capital requirement and a minimum liquidity requirement in the form of $-d_{IL} \geq \hat{d}_{IL}$, where the Central Planner chooses $\hat{d}_{IL}$ in the beginning of period zero. These two instruments will be sufficient for the policy maker to replicate the constrained Central Planner’s allocation. The larger $\chi$ is (the smaller the fiscal capacity is), the higher the optimal amount of liquidity transferred to the crisis state is ($\frac{\partial d_{IL}}{\partial \chi} > 0$). The reason is that larger fiscal capacity implies larger $k_0^{CP}$. Given that it is always the case that the period zero borrowing constraint binds in the high state for the Central Planner, the larger

³¹The parameters used to produce Graph 5 are $\gamma = 0.7, \alpha = 0.8, A = 0.8, a_{1h} = 1.5, a_{1l} = 0, \pi_h = .55, n = .5, \theta = .45, \eta = 1.5$. 

29
period zero investment is financed with larger ex-ante borrowing against the crisis state (or less money transferred to the crisis state). This logic explains why larger fiscal capacity implies smaller \( \bar{d}_{1l} \). The minimum liquidity requirement will be binding only for countries with fiscal capacity, \( \chi \), for which the dashed line is above the solid line in Graph 5. In that sense, no regulation will be required for countries with small fiscal capacity. However, conditional on the minimum liquidity requirement binding, \( \bar{d}_{1l} \) decreases as the fiscal capacity increases.

In practice, the minimum liquidity requirement is often defined as liquid assets over risky assets (which in the model is period zero investment, \( k_0 \)). The analogue in this model would be \( -\frac{\bar{d}_{1l}}{k_0} \geq \bar{d}_{rl} \) which optimally would be set to \( \bar{d}_{rl} = -\frac{\bar{d}_{CP}^l}{k_0^{CP}} \) (given that the optimal \( \bar{\rho} \) is set to \( \bar{\rho}^{CP} = \frac{n}{k_0^{CP}} \)). Plotting \( -\frac{\bar{d}_{CP}^l}{k_0^{CP}} \) one can see that countries with larger fiscal capacity should have lower minimum liquidity requirements defined as liquidity transferred to the low state over period zero investment. The intuition is based on the same argument as to why \( \frac{\partial \bar{d}_{1l}}{\partial \chi} > 0 \). Larger fiscal capacity implies larger \( k_0^{CP} \) and smaller \( \bar{d}_{CP}^l \).

![Graph 6](image)

In Graph 7, I replot Figure 5, but this time I keep \( \chi \) fixed and vary \( N \). Varying \( N \) does not change the Central Planner’s allocation but affects how strong the moral hazard is.
The smaller $N$ is, the larger the moral hazard is, and from Graph 7 one can see that the banker chooses to pledge too high of a payment in the crisis state, where $\rho$ is such that $k^{CE}_0 = k^{CP}_0$. Given that $d^{CP}_{1l}$ and $k^{CP}_0$ are not a function of $N$, the optimal minimum liquidity requirement, $\bar{d}_{1l} = -d^{CP}_{1l}$ (or $\bar{d}_{1l} = -\frac{d^{CP}_{1l}}{k^{CP}_0}$), will be just a constant and so will be the optimal minimum capital ratio $\rho = \frac{n}{k^{CP}_0}$. In this model, the strength of the moral hazard alone will not generate a role for differential regulation across countries, as long as the constrained Central Planner’s allocation is replicated. The intuition is related to the fact that the moral hazard appears only in the first order condition of the banker with respect to $k_0$ from the Ramsey problem, and this first order condition plays no role in determining the optimal minimum capital requirement or the minimum liquidity requirement.\footnote{One can also calculate the optimal $\rho$, conditional on no minimum liquidity requirement. For countries with large fiscal capacity and a few banks (strong moral hazard), the constrained Central Planner’s allocation can be no longer replicated. However, it is still the case that larger fiscal capacity implies lower $\rho$ (or even $\rho = 0$). If the policy maker cannot prevent the banker from borrowing first against the crisis state, the optimal allocation from the Ramsey Problem with only an ex-ante minimum capital requirement is to borrow to the maximum which is what the banker from the Ramsey Problem with no ex-ante regulation would optimally choose himself.}

What this result implies is that if one believed that the only source of ex-ante inefficiency was the moral hazard, conditional on the constrained Central Planner’s allocation being replicated, there would be no role for differential regulation across countries. This is in line with what the Basel Accord is currently promoting. However, given that fire sales have played an important role in financial crises, ignoring pecuniary externalities as another source of inefficiency would be an important omission. As I have shown in this paper, pecuniary externalities generate a role for differential cross country bank regulation, as a function of the fiscal capacity of the country.
7 A "Price" Instrument Versus A "Quantity" Instrument

In this section, I show that whether larger fiscal capacity implies more or less ex-ante regulation depends critically on the instrument used. I show that if the policy maker has an access to a "price" instrument such as a tax on period zero investment, the result is the opposite from the case where the instrument used is a "quantity" instrument such as a minimum bank capital requirement. If the regulatory instrument was a tax on period zero investment, larger fiscal capacity implies an optimally higher tax on period zero investment.

I solve for the Ramsey Problem with a finite number of banks using a tax on period zero investment instead of a minimum capital requirement. The only change in the set up of the Ramsey Problem is that the period zero budget constraint becomes

\[ k_0^i (1 + \tau^i_{k_0}) - n + T^i_{k_0} \leq \sum_s \pi_s d^i_s \quad [z^i_0] \] (15)

where \( \tau^i_{k_0} \) is the bank specific tax on period zero capital. The revenues from the tax are distributed equally back to the bankers where \( T^i_{k_0} = -\sum_{i=1}^{N} \frac{1}{N} k^i_{0} \tau^i_{k_0} \). The following proposition proves that a larger fiscal capacity implies a larger tax on period zero investment as long as \( 1 < N < \infty \) and the tax on period zero investment is constant if \( N \rightarrow \infty \) (the case with a continuum of banks).

**Proposition 9** Conditional on the policy maker having an access to two ex-ante instruments — a tax on period zero investment ("price" instrument), \( \tau_{k_0} \), and a minimum liquidity requirement, one can show that \( \tau_{k_0} > 0 \). If \( N \rightarrow \infty \) (no moral hazard) then \( \frac{\partial \tau_{k_0}}{\partial \chi} = 0 \). If \( 1 < N < \infty \) and \( A > 1 - \theta \) then \( \frac{\partial \tau_{k_0}}{\partial \chi} < 0 \). \( \tau_{k_0} \) and \( \frac{\partial \tau_{k_0}}{\partial \chi} \) are given by

\[ \tau_{k_0} = \left[ \frac{z^C P (k^T_{1l}) - z^C E,N (k^T_{1l}, \chi)}{z^C E,N (k^T_{1l}, \chi)} \right] \Phi > 0 \] (16)

\[ \frac{\partial \tau_{k_0}}{\partial \chi} = -\frac{\partial z^C E,N (k^T_{1l}, \chi)}{\partial \chi} \cdot \frac{z^CP (k^T_{1l}, \chi)}{z^C E,N (k^T_{1l}, \chi)} \] (17)

where \( \Phi > 0 \) is a constant given in the Appendix, Section 10.5.9.

**Proof of Proposition 9.** See Appendix, Section 10.5.9. ■

If the instrument of choice was a tax on period zero investment, equation 16 shows that the optimal tax is positive since the banker wants to overinvest relative to the Central Planner due to both the pecuniary externality and the moral hazard. This is captured by the fact that \( z^C P (k^T_{1l}) > z^C E,N (k^T_{1l}, \chi) \). \( ^{33} \)

\(^{33}\)See the Proof of Proposition 7 in the Appendix, Section 10.5.7.
One can show that the constrained Central Planner’s allocation can be achieved using either a tax on period zero investment or a minimum bank capital requirement (conditional on imposing also a minimum liquidity requirement if necessary). However, a "price" instrument is very different from a "quantity" instrument in the way it achieves the optimal allocation. By setting \( \tau_{k_0} \), the policy maker can no longer set directly the amount of \( k_0^{CE} \), which was true in the case of a minimum capital requirement. \( \tau_{k_0} \) affects the marginal cost of \( k_0 \) (or the "price" of \( k_0 \)), as perceived by the banker, which is why a tax instrument can be thought of as a "price" instrument. In equilibrium, if the ex-ante instrument is a tax, \( k_0 \) is determined by the first order condition of the banker with respect to \( k_0 \), which equates the marginal benefit of \( k_0 \) to the marginal cost of \( k_0 \).

The optimal \( \tau_{k_0} \) approximately equals the size of the overinvestment which is given by the scaled difference between the marginal benefit of \( k_0 \), as perceived by the banker, minus the marginal benefit of \( k_0 \) as perceived by the constrained Central Planner. This difference is approximately equal to 

\[
\frac{z_{ii}^{CE}(k_T) - z_{ii}^{CE,N}(k_T^{iN})}{z_{ii}^{CE,N}(k_T^{iN})}.
\]

The larger the difference in the perceived marginal benefit of \( k_0 \) is, the larger the tax on capital has to be, in order for the policy maker to be able to replicate the constrained Central Planner’s allocation. Due to the linearity assumption, the equilibrium \( k_T^{i} \) from the Central Planner’s problem does not vary with the fiscal capacity of the country (see equation 10). \( \chi \) does not enter directly the marginal benefit of \( k_0 \), as perceived by the Central Planner, since the Central Planner internalizes both the marginal cost and the marginal benefit of the bail-out and, in equilibrium, they cancel out. In contrast, \( \chi \) enters directly the marginal benefit of \( k_0 \), as perceived by the banker in the Ramsey Problem.\(^{34}\) Conditional on a finite number of banks, \( N < \infty \), the banker perceives the bail-out to be larger for countries with a larger fiscal capacity.\(^{35}\) Therefore, in order to achieve a given level of \( k_0 \), the policy maker will have to increase \( \tau_{k_0} \) by more for countries with larger fiscal capacity, since the perceived bail-out and, hence, the moral hazard in those countries are stronger. If one considers the case of \( N \rightarrow \infty \), \( \frac{\partial z^{CE,N}_{ii}(k_T^{iN})}{\partial \chi} = 0 \) because there is no moral hazard. In that case, \( \tau_{k_0} \) is still positive but it is no longer a function of the fiscal capacity.

In summary, the key reason why the size of the moral hazard affects the "price" instrument and not the "quantity" instrument is the following. The moral hazard enters into the model through the first order condition of the banker from the Ramsey Problem since the banker internalizes the benefit of the bail-out, but not the cost. If the ex-ante regulatory instrument is a tax on period zero investment, \( \tau_{k_0} \) is determined by combining the first order condition of the banker from the Ramsey Problem with respect to \( k_0 \), and the first order condition of the Central Planner with respect to \( k_0 \). In contrast, when the instrument is a minimum capital requirement, the first order condition of the banker with respect to \( k_0 \) from the Ramsey Problem will no longer play a role and, hence, the moral hazard does not affect the optimal minimum capital requirement. This is why, as long as the policy maker can replicate the constrained Central Planner’s allocation, the size of the moral

\(^{34}\) The first order condition of the banker with respect to \( k_0 \) from the Ramsey Problem is given by equation 81 in the Appendix.

\(^{35}\) Mathematically, this implies that \( \frac{\partial z^{CE,N}_{ii}(k_T^{iN})}{\partial \chi} > 0 \) and \( \frac{\partial z^{CE}_{ii}(k_T)}{\partial \chi} = 0 \).
hazard per se does not affect the optimal minimum capital ratio — a "quantity" instrument, but it affects the ex-ante tax on period zero investment — a "price" instrument.

8 Endogenous Equity — The Case With A Continuum of Banks

So far I assumed that the period zero equity in the model was fixed, and it was equal to the period zero assets of the banker, \( n \). In this section, I endogenize the equity choice. Bankers will try to circumvent the minimum capital requirement constraint by raising more equity. As a result, having only one ex-ante instrument in the form of a minimum capital requirement will be insufficient to replicate the constrained Central Planner’s allocation. Moreover, I show that when the model is extended to include costly equity, the pecuniary externalities provide a reason not only for an ex-ante regulation, but also for an ex-post regulation in the form of a minimum amount of equity raised requirement in a crisis. Finally, using simulations, I show that the key result, that countries with smaller fiscal capacity should have higher ex-ante minimum capital requirements, remains, even if the constrained Central Planner’s allocation can be no longer replicated.

I will consider the case with a continuum of banks, which shuts off the moral hazard. Therefore, in this section, I will focus only on the pecuniary externalities. So far in the model, borrowing was state contingent and collateralized, and the net cost of borrowing was zero given that the lenders are risk neutral. The banker was not able to pledge future cash flows and, more specifically, period two cash flow, \( A \), because he could run away with it, while he could not run away with the collateral.

In this section, I endow the consumer with a costly monitoring technology, where if the consumer uses this technology, he can prevent the banker from running away with the cash flow in \( t = 2 \). However, using this technology will be costly. In a way, this assumption is a proxy for the fact that if the consumer lends to the banker using a non-collateralized contract, such as an equity contract, he will have to invest more time, effort and money into ensuring that he will be repaid in the future, and he will have to be compensated for this cost by the borrower. Therefore, this costly borrowing contract resembles equity borrowing, which is why I treat it as equity when defining the minimum capital requirement. Furthermore, I assume that if the banker raises equity in \( t = 0 \), the equity holders can prevent the banker from closing the bank in \( t = 1 \). Therefore, the banker loses his ability to threaten the owners of state contingent debt that he will shut down the bank in \( t = 1 \) unless they accept the "take-it-or-leave-it" offer. As a result, in \( t = 0 \), the banker can borrow

\[\text{36 Alternatively I could have assumed that } A \text{ is stochastic and that there is asymmetric information between the banker and the consumer and only the banker observes } A. \text{ Then the monitoring technology would have been used to make sure that the banker is not lying regarding the realization of } A. \text{ This alternative specification is in the spirit of [Townsend, 1979] and [Bernanke & Gertler, 1989]. The results would be equivalent.} \]

\[\text{37 Markets are endogenously incomplete in this model, and it is a well known fact that it is hard to endogenously derive borrowing instruments that resemble the equity and debt contracts we observe in the real world. As a result, it is inevitable that one has to make a judgement call regarding which type of borrowing will be treated as equity and which will not. One of the most important properties of equity is that it is considered to be more costly than non-collateralized borrowing either because of a [Myers & Majluf, 1984] type of asymmetric information story or because of the story I propose here.} \]
against the value of the collateral in both periods one and two. For more details see the online Appendix.

Define the amount borrowed against the last period cash flow by banker $i$ as $\varepsilon_1^i$ if the borrowing is done in period zero, and as $\varepsilon_1^i$ if it is done in period 1, state $s$. From now on I will refer to $\varepsilon_0^i$ as the equity raised in period zero and $\varepsilon_1^i$ as the equity raised in period one state $s$. I will assume that the cost of monitoring is convex and given by $\sigma(\varepsilon_1^i)$ for the equity raised in state $s$ period one. The monitoring cost is given by $\sigma(\varepsilon_0^i)$ for the equity raised in period zero.\(^{38}\) Therefore, if the banker borrows $\varepsilon_0^i$ in period 1 state $s$, he will have to repay the consumer $\varepsilon_1^i + \sigma(\varepsilon_1^i)$ in period $t = 2$ state $s$. Similarly, if he borrows $\varepsilon_0^i$ in $t = 0$, with probability $\pi_h$ he will have to pay back $\varepsilon_0^i + \sigma(\varepsilon_0^i)$ in $t = 2$ in the high state and with probability $\pi_l$ he will have to pay $\varepsilon_0^i + \sigma(\varepsilon_0^i)$ in $t = 2$ in the low state, leading to an expected cost of period zero borrowing of $\varepsilon_0^i + \sigma(\varepsilon_0^i)$. I will consider only parametrization where there will be no default and $Ak_1^i > \varepsilon_0^i + \sigma(\varepsilon_0^i) + \varepsilon_1^i + \sigma(\varepsilon_1^i)$.\(^{39}\)

There are two reasons why I assume that the monitoring cost is convex with respect to the equity raised in the given period and state of nature. The first one is technical. Given that the production technology of the banker is linear, if the cost of monitoring was linear, it would lead to a corner solution and make the problem trivial. The second reason is that it is common in the corporate finance literature to model costly equity in a reduced form way by assuming a convex cost of raising equity.\(^{40}\) I could have made the monitoring technology state contingent but, for simplicity, I will consider it to be the same regardless of whether the monitoring takes place in the high or the low state.

First, I solve for the constrained Central Planner’s allocation. Details of the solution are provided in the online Appendix.

**Proposition 10** Conditional on parametrization such that $q_U < 1$ and $q_M = 1$, considering a symmetric equilibrium for the Central Planner’s Problem with costly equity and no commitment, the equilibrium is unique and exists and is one of the following types: 1) $z_{0}^{CP,E} = z_{1l}^{CP,E} > z_{1h}^{CP,E}$ (a interior equilibrium) and 2) $z_{0}^{CP,E} > z_{1s}^{CP,E}$ (a corner equilibrium where the banker borrows to the maximum in $t = 0$). The optimal bail-out is pinned down by equation 9. $\varepsilon_0$ and $\varepsilon_1$ are determined by

$$1 + \sigma'(\varepsilon_0) = z_{0}^{CP,E}$$
$$1 + \sigma'(\varepsilon_1) = z_{1s}^{1,CP,E}$$

\(^{38}\)I assume that the banker borrows the whole amount from a single consumer and because consumers are risk neutral, they receive a net expected return of zero.

\(^{39}\)Alternatively, I could have assumed that in $t = 0$ the banker can write separate state contingent contracts against the high and the low state cash flows in $t = 2$. Changing the model in such a way is unlikely to affect the key results.

\(^{40}\)Also having the cost be convex with respect to the amount of equity raised in each period and state implies that it will be always optimal to raise equity in every period and state of nature. This assumption can be thought of as a proxy for the fact that in a [Myers & Majluf, 1984] environment with asymmetric information, where raising equity sends a signal that the bank is in trouble, the consumers would demand larger marginal payment from the bankers that raise more equity at once.
If the equilibrium is of Type 1, \( z_{CP;E}^0 \) is given by equation 10. \( z_{CP;E}^1 \) and \( z_{CP;E}^1 \) are given by equations 12 and 11. Also \( z_{CP;E}^1 = z_{0,CP;E}^1 \).

Proof of Proposition 10. See online Appendix.

As before one can show that the constrained Central Planner values wealth more in the crisis state than in the high state and, as a result, there is only one feasible interior equilibrium (an equilibrium of Type 1) conditional on a fire sale in the crisis state. The only difference between Proposition 2 and Proposition 10 is that now the Central Planner chooses the optimal amount of equity raised as well, where the marginal cost of raising one more unit of equity, \( 1 + \sigma'(\varepsilon_{1s}) \) is equated to the marginal benefit given by how much the Central Planner values wealth in the hands of the banker in \( t = 1 \), state \( s \), \( z_{1s;CP;E}^1 \). There is a similar equation for the equity raised in period \( t = 0 \).

When comparing the allocation from the Central Planner and the allocation from the Ramsey Problem, where there is no ex-post instrument besides the optimal bail-out, one can show that

**Corollary 2** In a crisis, the banker will choose to raise less equity than the constrained Central Planner \( \varepsilon_{1l}^{CE,E} < \varepsilon_{1l}^{CP,E} \).

Proof of Corollary 2. See online Appendix.

Similarly to the case with no equity, in the crisis state the banker values wealth less than the Central Planner does since \( z_{1l;CE,E}^1 = \frac{A + 1 - \theta}{F(k_{1l}^0) - \theta} < z_{1l;CP,E}^1 \) and the equity raised by the banker in the competitive equilibrium in the low state in \( t = 1 \) is determined by \( 1 + \sigma'(\varepsilon_{1l}) = z_{1l;CE,E}^1 \). Given that \( z_{1l;CE,E}^1 < z_{1l;CP,E}^1 \) and due to the convexity of \( \sigma(\cdot) \), then \( \varepsilon_{1l}^{CE} < \varepsilon_{1l}^{CP} \).

The intuition for the result in Corollary 2 is based on the presence of pecuniary externalities. The reason why the bankers want to raise too little equity relative to what is socially optimal is because they do not internalize the fact that the more equity is raised, the lower the fire sale will be. This will relax the budget constraints of the other bankers and will reduce the transfer of resources from the more productive to the less productive sector.

Define the minimum capital requirement as \( \rho k_0 \leq n + \varepsilon_0 \) and the minimum equity raised requirement during a crisis as \( \varepsilon_{1l} \geq \tilde{\varepsilon}_{1l} \). Setting \( \rho = \frac{n + \varepsilon_{0,CP,E}^1}{k_0^{CP,E}} \) is no longer sufficient to replicate the constrained Central Planner’s allocation (even if \( \varepsilon_{1l} = \varepsilon_{1l}^{CP,E} \)), given that the banker can choose to raise equity larger than \( \varepsilon_{0,CP,E}^1 \), in order to be able to overinvest relative to the Central Planner. Another ex-ante instrument such as a maximum requirement on period zero investment will be necessary, in order to replicate the constrained Central Planner’s allocation. However, one does not observe such a regulatory instrument in practice, and the focus of this paper is only on instruments.

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41For the solution of the Ramsey Problem see the online Appendix.
that have been employed in practice. Therefore, I simulate the Ramsey Problem with only a single ex-ante instrument in the form of a minimum bank capital requirement and an ex-post minimum equity raised requirement. The results for the optimal \( \rho \) are presented below.\(^{42}\)

In Graph 8, \( \rho^{RP} \) is the optimal minimum capital ratio if there is only a single ex-ante instrument. \( \rho^{CP} = \frac{n + \psi^{CP, E}}{k_{0}^{E}} \) is what the optimal \( \rho \) would be if there was also a second ex-ante instrument such as a requirement on maximum period zero investment, which will be sufficient to replicate the constrained Central Planner’s allocation. Notice that \( \rho^{RP} > \rho^{CP} \). The intuition for the result is that given that bankers are tempted to raise more equity than socially optimal in order to circumvent the minimum capital constraint, the policy maker has to impose a more stringent minimum capital requirement.

\(^{42}\)In the simulations presented I assume a quadratic cost of raising equity \( \sigma(\varepsilon) = \nu \varepsilon^2 \) where the parameters used to produce the graph are \( \gamma = 0.7, \alpha = 0.8, A = 0.85, a_{1h} = 1.47, a_{1l} = 0, \pi_{h} = .55, n = .7, \theta = .2, \eta = 1.5, \nu = 1.5 \)
Graph 9 shows that despite the fact that $\rho^{RP} > \rho^{CP}$, the banker still overinvests relative to what the Central Planner would want to invest optimally. Most importantly, the key result, that countries with smaller fiscal capacity should have larger minimum capital requirements relative to countries with larger fiscal capacity, remains robust when costly equity is introduced, even though the constrained Central Planner’s allocation can be no longer replicated.

9 Further Discussion and Conclusion

The key result derived in this paper is that cross country bank regulation should not be synchronized given the heterogeneity across countries in terms of fiscal capacity. This paper provides a normative result — countries with larger fiscal capacity should have lower minimum capital requirements relative to countries with smaller fiscal capacity. The comparative static is a function only of the fiscal capacity of a country which is readily observable given that one can measure variables such as the size of the banking sector relative to GDP and the size of sovereign debt. As a result, the result is fairly information insensitive assuming that there are not large differences across countries regarding the other parameters of the model. The result seems also robust to introducing costly equity and moral hazard.

One could argue that countries are heterogeneous in many other dimensions, in addition to fiscal capacity. For example, some countries like Switzerland, which would be financially constrained, according to the definition of this paper, have independent monetary policy and can print money in order to bail-out the banks, in addition to taxing and borrowing. Others, like Greece and Spain, which are also currently fiscally constrained, do not have an independent monetary policy due to being members of the European Monetary Union. While a richer model will be required to study the nuances of optimal bank regulation if monetary policy is introduced as a second instrument, this model provides some intuition of what the result might be. In most standard monetary policy models, the cost of printing money can be reduced to a convex cost of inflation. Therefore, countries with an independent monetary policy will be able to provide larger bail-outs relative to countries without independent monetary policy, all else equal, since they have an access to two (not one) policy instruments the deadweight loss of which is convex – printing money and taxing/borrowing. Hence, they will optimal use both instruments to bail-out the banks. In the framework of the model presented in this paper, if two countries are exactly the same but one has an independent monetary policy and the other does not, one can think of the country with an independent monetary policy as a less fiscally constrained country, which should have a lower minimum capital requirement.

Finally, one can make the argument that one of the main reasons why Basel was implemented and promoted synchronized bank regulation across countries was to introduce a "level playing" field. This model is a legacy model and does not consider the dynamics of what might happen if I were to introduce heterogeneous regulation. It does not address the questions whether banks will relocate

43 Technically printing money is another form of taxing — an "inflationary" tax.
to countries with laxer regulation and how this will affect the welfare of the country they leave. Conditional on assuming that there are large fixed costs to banks relocating either because of fixed investment in human capital and buildings or due the fact that markets are naturally segmented due to the value of being close to the borrower in order to be able to have lower monitoring cost as a bank, most of the results in this paper will remain. Fiscally constrained countries might not choose to raise minimum capital requirements significantly higher than less fiscally constrained countries because they take into account that if the benefit of relocating is high enough for the bankers, they will be willing to pay the fixed cost and relocate. However, most likely these fixed costs are quite high, which is why, there will be still a substantial leeway for differential regulation across countries.

References


10 Appendix

The functional forms used for the simulations are:

The functional form of the consumer production technology in $t = 1$ is given by

$$F (k^T) = \frac{(1 - \gamma)}{\alpha} - \frac{(1 - \gamma) e^{-\alpha k^T}}{\alpha} + \gamma k^T > 0$$

where $0 < \alpha$ is a parameter that controls the concavity of the production technology and $0 < \gamma < 1$ is the refinancing cost. The larger $\alpha$ is, the smaller $q_{1l}$ is for a given $k^T$. This functional form guarantees that the assumptions made regarding $F (\cdot)$ are satisfied

$$q_{1l} = F' (k^T) = (1 - \gamma) e^{-\alpha k^T} + \gamma > 0 \text{ where } 1 > \gamma \geq 0$$

$$F'' (k^T) = -\alpha (1 - \gamma) e^{-\alpha k^T} < 0$$

$$F''' (k^T) = \alpha^2 (1 - \gamma) e^{-\alpha k^T} > 0$$

$$F (0) = 0; F' (0) = 1; \lim_{k^T \to \infty} F' (k^T) = \gamma$$

The functional form for the deadweight loss from the bail-out is $\delta (B_t, \chi) = \chi B_t^\eta$ and the functional for for the convex cost of raising equity is $\sigma (\varepsilon) = \nu \varepsilon^2$

Define

$$\tilde{F} (k_{1s}^T) = \begin{cases} F (k_{1s}^T) \text{ if } k_{1s}^T \geq 0 \\ k_{1s}^T \text{ if } k_{1s}^T \leq 0 \end{cases}$$

10.1 The Problem of the Consumer

The endowment of the consumer is given by $e$ and the consumer receives it in every period $t$ and state $s$. I solve the problem of the consumer backwards. In period 2 the banker maximizes $\max_{k_{2s}^T} (k_{2s}^T - q_{2s} k_{2s}^T)$ taking $q_{2s}$ as given, which implies that $q_{2s} = 1$. In period 1, state $s$, the consumer chooses the amount to invest in the production technology, $k_{1s}^T$ where the price of capital is $q_{1s}$. He also chooses how much of the period 2 state $s$ asset to buy, $d_{2s}$, where the price of this asset is $p_{2s}$ and it pays one unit of the consumption good in $t = 2$, state $s$. Since each banker is small, he takes all the prices as given. In $t = 1$, state $s$, the representative consumer maximizes

$$\max_{k_{1s}^T, d_{2s}} \left[ 2e + d_{1s} + p_{2s} d_{2s} + d_{2s} + F (k_{1s}^T) - q_{1s} k_{1s}^T \right]$$

where $d_{1s}$ is a state variable which is the amount the banker promised to pay the consumer in $t = 1$ state $s$. The first order condition with respect to $k_{1s}^T$ pins down the equilibrium price of capital in $t = 1$ as a function of the amount of "fire-sold" capital, $q_{1s} = F' (k_{1s}^T)$. The first order condition
with respect to $d_{2s}$ is an Euler equation which implies $p_{2s} = 1$ since consumers are risk neutral and their marginal utility of consumption is 1.

After plugging in for the consumer’s first order conditions in $t = 1$ and $t = 2$, the $t = 0$ optimization problem of the consumer is given by

$$\max_{d_{1s}} \left[ 3c - \sum_s \pi_s p_{1s} d_{1s} + \sum_s \pi_s \left( d_{1s} + F \left( k_{1s}^T \right) - q_{1s} k_{1s}^T \right) \right]$$

The consumer chooses the amount of state contingent asset, $d_{1s}$, to purchase where he takes the price of the asset $\pi_s p_{1s}$ as given and the asset pays one unit of the consumption good in state $s$ which occurs with probability $\pi_s$. The first order condition with respect to $d_{1s}$ implies $p_{1s} = 1$. Notice that since $p_{1s} = 1$ and $p_{2s} = 1$, consumers will be indifferent how much of the state contingent asset to purchase in period 1 and 2.

### 10.2 Ramsey Problem with a Continuum of Banks – No Commitment

Assume that the policy maker has an access to an ex-post bail-out and an ex-ante minimum capital requirement. I solve the model backwards. At the end of period $t = 1$, the banker chooses $\{k_{1s}^i, d_{2s}^i\}$ taking as given $\{B^i_s, k_0^i, d_{1s}^i, p^i \}$ and $q_{1s}, p_{1s}, p_{2s}$. Banker $i$ maximizes the dividend payments in the last period

$$\max_{k_{1s}^i, d_{2s}^i} (A + 1) k_{1s}^i - d_{2s}^i$$

subject to the collateral constraint of the banker in $t = 1$

$$d_{2s}^i \leq \theta k_{1s}^i \left[ \lambda_{2s}^i \right] .$$

Banker $i$ also takes into account the period one budget constraint

$$k_{1s}^i q_{1s} + d_{1s}^i \leq (q_{1s} + a_{1s} - \gamma) k_{0}^i + B^i_s + p_{2s} d_{2s}^i$$

where in equilibrium $q_{1s} = F'(k_{1s}^T)$ where $k_{1s}^T = \int_0^1 k_{1s}^T di$ and $k_{1s}^T = \max \{0, k_0^i - k_{1s}^i \}$ and $p_{1s} = \pi_s, p_{2s} = 1$. The Lagrangians are in square brackets. The first order condition with respect to $k_{1s}^i$ pins down the marginal value of wealth in period $t = 1$ state $s$ as perceived by the banker in $t = 1$

$$z_{1s}^{i,1} = \frac{A + 1 + \lambda_{2s}^i \theta}{q_{1s}} > 1$$

Since $A > 0$ and $q_{1s} \leq 1$, it will be always the case that the marginal value of wealth of the bankers in $t = 1$ as perceived by the bankers is greater than one. The first order condition with respect to $d_{2s}^i$ is given by

$$p_{2s} z_{1s}^{i,1} = 1 + \lambda_{2s}^i$$

Since $z_{1s}^{i,1} > 1$ and in equilibrium $p_{2s} = 1$, then $\lambda_{2s}^i = \left( z_{1s}^{i,1} - 1 \right) > 0$ and the banker chooses to borrow to the maximum in $t = 1$ using all of $k_{1s}^i$ as a collateral. I can also re-write $z_{1s}^{i,1}$ as

$$z_{1s}^{i,1} = \frac{A + 1 - \theta}{q_{1s} - \theta}$$
At the beginning of $t = 1$ in the low state, the policy maker chooses $B_l$ taking into account the optimal allocation chosen by the banker at the end of $t = 1$, \( \{ k_{1l}^i (B_l^i), d_{2l}^i (B_l^i) \} \), and that the equilibrium is symmetric. The difference between the banker and the policy maker is that the policy maker internalizes the fact that the price is a function of $k_{1l}^T$. The policy maker takes into account the budget constraint in the low state which pins down $k_{1l}$ (the Lagrangian of which is $z_{1l}^{1,RP}$) and also that $d_{2s} = \theta k_{1s}$ and $p_{2s} d_{2s} = d_{2s}$

\[
2e - (\delta (B_l) + B_l) + F (k_{1l}^T) - F' (k_{1l}^T) k_{1l}^T + d_{1l} + d_{2l} - p_{2l} d_{2l} + (A + 1) k_{1l} - d_{2l}
\]

subject to the budget constraint

\[
k_{1l} \left( \bar{F}' (k_{1l}^T) - \theta \right) + d_{1l} \leq \left( \bar{F}' (k_{1l}^T) + a_{1l} - \gamma \right) k_0 + B_l \quad \left[ z_{1l}^{1,RP} \right]
\]

First order condition with respect to $k_{1l}$:

\[
z_{1l}^{1,RP} = \frac{\bar{F}'' (k_{1l}^T) k_{1l}^T + A + 1 - \theta}{\bar{F}'' (k_{1l}^T) k_{1l}^T + \bar{F}' (k_{1l}^T) - \theta}
\] (18)

First order condition with respect to $B_l$:

\[
1 + \delta' (B_l) = z_{1l}^{1,RP}
\] (19)

where $\delta' (B_l)$ is a function of the aggregate bail-out and $z_{1l}^{1,RP}$ is a function of the aggregate fire sale of capital in the low state.

At the end of $t = 0$, banker $i$ chooses $\{ k_{0s}^i, d_{1s}^i \}$ taking as given $\rho^i$ and prices $p_{1s}, q_{1s}$. Also he takes into account that the optimal bailout is not a function of his choice variables since the banker is too small to affect the aggregate fire sale. Finally, banker $i$ takes into account that the allocation she chooses at the end of $t = 1$ is a function of period zero choice variables. I plug in $d_{2s}^i$ and take into account that $k_{1s}^i$ is pinned down by the period one budget constraint while $B_l^i$ is pinned down by the first order condition of the policy maker in the beginning of period $t = 1$. The banker maximizes ex-ante welfare

\[
\max_{k_{0s}^i, d_{1s}^i, k_{1s}^i} \sum_s \pi_s (A + 1 - \theta) k_{1s}^i
\]

subject to the period one and period zero budget constraints

\[
k_{1s}^i (q_{1s} - \theta) + d_{1s}^i \leq (q_{1s} + a_{1s} - \gamma) k_{0s}^i + B_s^i \quad \left[ \pi_s z_{1s}^{i,0} \right]
\]

\[
k_{0s}^i - n \leq \sum_s \pi_s p_{1s} d_{1s}^i \quad \left[ z_{0s}^i \right]
\]

The period one collateral constraint

\[
d_{1s}^i \leq \theta (q_{1s} - \gamma) k_{0s}^i \quad \left[ \pi_s \lambda_{1s}^i \right]
\]

43
and the minimum capital requirement constraint

$$\rho^i k_0^i \leq n$$

First order condition with respect to $k_{i,0}$

$$\sum_s \pi_s z_{1s}^0 (q_{1s} + a_{1s} - \gamma) - z_0^i - \rho^i t^i + \sum_s \pi_s \lambda_{1s}^i (q_{1s} - \gamma) = 0$$

First order condition with respect to $d_{1s}^i$

$$-z_{1s}^0 + p_{1s} z_{1s}^i - \lambda_{1s}^i = 0$$

First order condition with respect to $k_{1s}^i$

$$z_{1s}^0 = \frac{A + 1 - \theta}{q_{1s} - \theta}$$

### 10.3 Constrained Central Planner’s Problem – No Commitment

I solve the problem backwards. The Central Planner is subject to exactly the same constraints as the banker in the CE but he takes into account that the equilibrium played is symmetric and his actions affect aggregate prices – $p_{1s}, p_{2s}$ and $q_{1s}$. Since the equilibrium asset prices are pinned down to 1, $p_{1s} = p_{2s} = 1$, the only relevant price is the price of capital in the middle period $q_{1s} = \tilde{F}' (k_{1s}^T)$. I set $B_h = 0$ in equilibrium since I assumed that the cost of the bail-out is prohibitively high during normal times – period zero and period one in the high state. Taking into account that in equilibrium $p_{2s} = p_{1s} = 1$, the period one optimization problem of the Central Planner, given the state variables, $\{k_0, d_{1s}\}$, is

$$\max_{k_{1s}, d_{1s}, d_{2s}} 2c - (\delta (B_s) + B_s) + F (k_{1s}^T) - \tilde{F}' (k_{1s}^T) k_{1s}^T + d_{1s} + (A + 1) k_{1s} - d_{2s}$$

The optimization problem is subject to the collateral constraint in $t = 2$

$$d_{2s} \leq \theta k_{1s} \quad \left[ \lambda_{2s}^{CP} \right].$$

The optimization problem in $t = 1$ is also subject to the period one budget constraint

$$k_{1s} \tilde{F}' (k_{1s}^T) + d_{1s} \leq \left( \tilde{F}' (k_{1s}^T) + a_{1s} - \gamma \right) k_0 + B_s + d_{2s} \quad \left[ z_{1s}^{1,CP} \right]$$

From the first order condition with respect to $k_{1s}$

$$z_{1s}^{1,CP} = \frac{\tilde{F}'' (k_{1s}^T) k_{1s}^T + A + \lambda_{2s}^{CP} \theta}{\tilde{F}' (k_{1s}^T) + \tilde{F}'' (k_{1s}^T) k_{1s}^T}$$

The first order condition with respect to $d_{2s}$ is

$$z_{1s}^{1,CP} = 1 + \lambda_{2s}^{1,CP} \geq 1 \quad (20)$$
First order condition with respect to \( B_{1s} \)

\[
z_{1s}^{1CP} = 1 + \delta' (B_s)
\]

First I prove that \( \lambda_{2s}^{1CP} > 0 \). Since \( p_{1s} \leq 1 \) and \( A + 1 + \lambda_{2s}^{1CP} \theta > 1 \) then \( z_{1s}^{1CP} > 1 \). From equation 20 \( \lambda_{2s}^{1CP} = z_{1s}^{1CP} - 1 > 0 \) which completes the proof that \( \lambda_{2s}^{1CP} > 0 \). Hence \( d_{2s} = \theta k_{1s} \). Re-writing the first order condition with respect to \( k_{1s} \) using the fact that \( \lambda_{2s}^{1CP} = z_{1s}^{1CP} - 1 \)

\[
z_{1s}^{1CP} = \frac{\tilde{F}'' (k_{1s}^T) k_{1s}^T + A + 1 - \theta}{\tilde{F}' (k_{1s}^T) + \tilde{F}'' (k_{1s}^T) k_{1s}^T - \theta}
\]

(21)

The Central Planner’s optimization problem in \( t = 0 \) becomes

\[
\max_{k_0, \{k_{1s}, d_{1s}\}_{s=1,h}} 3e - \sum \pi_{1s} d_{1s} + \sum \pi_s \left[ d_{1s} - d_{2s} + d_{2s} + F (k_{1s}^T) \right]
\]

\[
- \tilde{F}' (k_{1s}^T) k_{1s}^T - B_s - \delta (B_s) + (A + 1) k_{1s} - d_{2s}
\]

Using the fact that \( d_{2s} = \theta k_{1s} \) and taking into account that in equilibrium \( p_{2s} = 1 \) and \( p_{1s} = 1 \).

\[
\max_{k_0, \{k_{1s}, d_{1s}\}_{s=1,h}} 3e + \sum \pi_s \left[ \tilde{F} (k_{1s}^T) - \tilde{F}' (k_{1s}^T) k_{1s}^T - B_s - \delta (B_s) + (A + 1 - \theta) k_{1s} \right]
\]

The optimization problem is subject to the following constraints. The budget constraint in period zero

\[
k_0 \leq n + \sum \pi_s d_{1s} \quad \left[ \pi_s z_{1s}^{0CP} \right]
\]

and the period one collateral constraint

\[
d_{1s} \leq \theta \left( \tilde{F}' (k_{1s}^T) - \gamma \right) k_0 \quad \left[ \pi_s \lambda_{1s}^{CP} \right]
\]

(22)

First order condition with respect to \( k_0 \)

\[
\sum \pi_s \left( \begin{array}{c}
- \tilde{F}'' (k_{1s}^T) k_{1s}^T - \frac{\partial B_s}{\partial k_0} (1 + \delta' (B_s)) \\
+ \lambda_{1s}^{CP} \theta \left[ \tilde{F}' (k_{1s}^T) - \gamma + \tilde{F}'' (k_{1s}^T) k_{1s}^T \right]
\end{array} \right) = z_{1s}^{CP}
\]

(23)

where \( \frac{\partial B_s}{\partial k_0} = \frac{1}{\delta' (B_s)} \frac{\partial \lambda_{1s}^{1CP}}{\partial k_{1s}^T} \)

First order condition with respect to \( k_{1s} \)
\[
\tilde{F}'' (k_{1s}^T) k_{1s}^T - \frac{\partial B_s}{\partial k_{1s}} (1 + \delta' (B_s)) + A + 1 - \theta
\]
\[+ z_{1s}^{0,CP} \left[ - \tilde{F}'' (k_{1s}^T) k_{1s}^T + \frac{\partial B_s}{\partial k_{1s}} - \left( \tilde{F}' (k_{1s}^T) - \theta \right) \right] - \lambda_{1s}^{CP} \theta \tilde{F}'' (k_{1s}^T) k_0 = 0 \quad (24a)\]

where \( \frac{\partial B_s}{\partial k_{1s}} = - \frac{1}{\delta''(B_s)} \frac{\partial z_{1s}^{1,CP}}{\partial k_{1s}} \)

First order condition with respect to \( d_{1s} \)
\[ z_0^{CP} - z_{1s}^{0,CP} - \lambda_{1s}^{CP} = 0 \]

10.4 Ramsey Problem with N Banks – No Commitment

Assume that the policy maker has an access to an ex-post bail-out, an ex-ante minimum capital requirement and an ex-ante minimum liquidity requirement. I solve the problem backwards assuming that debt markets re-open in \( t = 1 \). Averages are defined as \( k_{1s}^T = \sum_{i=1}^{N} \frac{1}{N} k_{is}^T \) and \( B_t = \sum_{i=1}^{N} \frac{1}{N} B_{is}^i \). Will take into account that \( q_{2s} = 1 \) and \( p_{2s} = p_{1s} = 1 \).

At the end of period \( t = 1 \), the banker chooses \( \{k_{1s}^i, d_{2s}^i\} \) taking as given \( \{B_{is}^i, k_{0}^i, d_{1s}^i\} \) and the period zero policy instruments. The banker no longer takes prices as given. The period one optimization problem of banker \( i \) is

\[
\max_{k_{1s}^i, d_{2s}^i} (A + 1) k_{1s}^i - d_{2s}^i
\]
subject to the collateral constraint in \( t = 1 \)
\[ d_{2s}^i \leq \theta k_{1s}^i \quad [\lambda_{2s}^i]. \]

The optimization problem is also subject to the period one budget constraint
\[ k_{1s}^i \tilde{F}' (k_{1s}^T) + d_{1s}^i \leq \left( \tilde{F}' (k_{1s}^T) + a_{1s} - \gamma \right) k_{0}^i + B_{is}^i + d_{2s}^i \quad [z_{1s}^{1,i}]. \]

First order condition with respect to \( k_{1s}^i \)
\[ z_{1s}^{1,i} = \frac{A + 1 + \lambda_{2s}^i \theta}{\tilde{F}' (k_{1s}^T) + \frac{1}{N} \tilde{F}'' (k_{1s}^T) k_{1s}^T} > 1 \]

First order condition with respect to \( d_{2s}^i \)
\[-1 - \lambda_{2s}^i + z_{1s}^{1,i} = 0 \]

Next I prove that \( \lambda_{2s}^i > 0 \). Since \( z_{1s}^{1,i} > 1 \), \( \lambda_{2s}^i = z_{1s}^{1,i} - 1 > 0 \).
\[ d_{2s}^i = \theta k_{1s}^i \quad (25) \]
In the beginning of period \( t = 1 \), in the low state, the policy maker chooses \( B_i^t \) given \( \{k_i^0, d_{1s}^i\} \) and the period zero policy instruments. The optimization problem is given by

\[
\max_{k_{1t}^i, B_i^t} 2e + \tilde{F}'(k_{1t}^T) - \tilde{F}'(k_{1t}^T) k_{1t}^T - \delta(B_i^t) + d_{1t}^i
\]

\[+ \sum_{i=1}^{N} \frac{1}{N} \left[ (A + 1 - \theta) k_{1t}^i - B_i^t + z_{1t}^{i,1,RP} \left( \left( \tilde{F}'(k_{1s}^T) + a_{1s} - \gamma \right) k_0^i + B_s^i + \theta k_{1s}^i - k_{1s}^i \tilde{F}'(k_{1s}^T) - d_{1s}^i \right) \right]
\]

First order condition with respect to \( k_{1t}^i \)

\[
\left( \tilde{F}''(k_{1t}^T) k_{1t}^T + A + 1 - \theta \right) + z_{1t}^{i,1,RP} \left( \theta - \tilde{F}'(k_{1t}^T) \right) = \tilde{F}''(k_{1t}^T) \sum_{j=1}^{N} \frac{1}{N} z_{1t}^{j,1,RP} k_{1t}^j
\]

Equation 26 holds for every \( i \) then \( z_{1t}^{i,1,RP} \) is the same for every \( i \) and one can simplify equation 26 as

\[
z_{1t}^{i,1,RP} = z_{1t}^{1,RP} = \frac{\tilde{F}''(k_{11t}) k_{11t}^T + A + 1 - \theta}{\tilde{F}''(k_{11t}) k_{11t}^T + \tilde{F}'(k_{11t}) - \theta}
\]

The first order condition with respect to \( B_i^t \) is

\[1 + \delta'(B_i^t) = z_{1t}^{1,RP}
\]

Since the policy maker is indifferent whom to give the bail-out to I will assume a symmetric equilibrium in order to solve the model, i.e. \( B_i^t = B_t \). At the end of period \( t = 0 \), the banker chooses \( \{k_i^0, d_{1s}^i\} \) taking as given period zero policy instruments and internalizing his effect on \( B_i^t \) and on his own future actions. I plug in for \( d_{1s}^i, d_{2s}^i \) and take into account that \( k_{1s}^i \) is pinned down by the period one budget constraint while \( B_i^t \) is pinned down by the first order condition of the policy maker with respect to \( B_i^t \) in the beginning of period \( t = 1 \). One can re-write the optimization problem above as

\[
\max_{k_{0s}^i, d_{1s}^i, k_{1s}^i} \sum_s \pi_s (A + 1 - \theta) k_{1s}^i
\]

subject to

\[
k_{1s}^i \left( \tilde{F}'(k_{1s}^T) - \theta \right) + d_{1s}^i \leq \left( \tilde{F}'(k_{1s}^T) + a_{1s} - \gamma \right) k_0^i + B_s^i \left[ \pi_s z_{1s}^{i,0} \right]
\]

\[
k_{0}^i - n \leq \sum_s \pi_s d_{1s}^i \left[ z_{1s}^{i} \right]
\]

\[
d_{1s}^i \leq \theta \left( \tilde{F}'(k_{1s}^T) - \gamma \right) k_0^i \left[ \pi_s \lambda_{1s}^i \right]
\]

\[
\rho^i k_0^i \leq n \left[ c^i \right]
\]

and finally subject to the minimum liquidity constraint
First order condition with respect to $k_{i,0}$

\[
-d_{iI}^t \geq \bar{d}_{iI}^t \quad [\pi t \varphi^i]
\]

\[
\sum_s \pi_s z_{i,s}^i \left( \tilde{F}'(k_{1s}^T) + a_{1s} - \gamma + \frac{1}{N} \tilde{F}''(k_{1s}^T) k_{1s}^T + \frac{\partial B^i_s}{\partial k_{0}^i} \right)
\]

\[
-z_{i}^i - \rho' i^i + \sum_s \pi_s \lambda_{1s}^i \theta \left( \tilde{F}'(k_{1s}^T) - \gamma + \frac{1}{N} \tilde{F}''(k_{1s}^T) k_{0}^i \right) = 0
\]

First order condition with respect to $k_{1s}^i$

\[
A + 1 - \theta + \tilde{z}_{1s}^i \left[ -\frac{1}{N} \tilde{F}''(k_{1s}^T) k_{1s}^T + \frac{\partial B_s^i}{\partial k_{1s}^i} - \left( \tilde{F}'(k_{1s}^T) - \theta \right) \right]
\]

\[
+ \lambda_{1s}^i \theta \left[ -\frac{1}{N} \tilde{F}''(k_{1s}^T) k_{0}^i \right] = 0
\]

First order condition with respect to $d_{iI}^t$

\[
-d_{iI}^t + z_{i}^i - \varphi^i = 0
\]

First order condition with respect to $d_{iI}^h$

\[
-d_{iI}^h + z_{0}^i - \lambda_{1h}^i = 0
\]

where \( \frac{\partial B_s^i}{\partial k_{0}^i} = \frac{1}{\delta(B_s^i) N} \frac{\partial z_{1I}^{RP}}{\partial k_{1I}^i} ; \frac{\partial B_s^i}{\partial k_{1I}^i} = -\frac{1}{\delta(B_s^i) N} \frac{\partial z_{1I}^{RP}}{\partial k_{1I}^i} \)

10.5 Proofs of Lemmas and Propositions

10.5.1 Proposition 1

**Proposition 1** Given Assumptions 1-5, considering a symmetric equilibrium for the Ramsey Problem with no commitment and assuming no ex-ante regulation $\rho = 0$, there is never a fire sale in the high state, $q_{1h} = 1$. Given the additional assumption

\[
(d')^{-1} \left( \frac{A}{1-\theta} \right) < \frac{(\gamma - \theta - a_{1I} + \theta (1-\gamma)) n}{1-\theta (1-\gamma)}, \quad \text{(Assumption 6)}
\]

it is always the case that there is a fire sale in the low state, $q_{1l} < 1$. The equilibrium exists and is unique and is one of the following types: 1) $z_{0}^{CE} = z_{1l}^{CE} > z_{1h}^{CE}$ (interior equilibrium) and 2) $z_{0}^{CE} > z_{1s}^{CE}$ (corner equilibrium where the banker borrows to the maximum in $t = 0$) where $z_{1s}^{CE} = z_{1s}^{0,CE}$. The optimal bail-out is pinned down by

\[
1 + \delta' (B_s^i) = z_{1I}^{1,RP} = \frac{\tilde{F}''(k_{1I}^T) k_{1I}^T + A + 1 - \theta}{\tilde{F}''(k_{1I}^T) k_{1I}^T + \tilde{F}'(k_{1I}^T) - \theta}
\]
If also the following assumption is satisfied

\[ 1 + \pi_l (\gamma - a_{1l} - F'(k_{1l}^T)) > \pi_h \left\{ \frac{(F'(k_{1l}^T) - \theta)}{(1 - \theta)} a_{1h} + F'(k_{1l}^T) (1 - \gamma) \right\} \]  

(Assumption 7)

where \( k_{1l}^T \) is pinned down by

\[
\frac{\left( \theta + a_{1l} - \gamma - \theta \left( \tilde{F}'(k_{1l}^T) - \gamma \right) \right) n}{1 - \pi_l \theta (1 - \gamma) - \pi_l \theta \left( \tilde{F}'(k_{1l}^T) - \gamma \right)} + (\delta')^{-1} \left( \frac{A + 1 - \tilde{F}'(k_{1l}^T)}{F''(k_{1l}^T) k_{1l}^T + \tilde{F}'(k_{1l}^T) - \theta} \right) + k_{1l}^T \left( \tilde{F}'(k_{1l}^T) - \theta \right) = 0
\]

(33)

the only possible equilibrium is the interior equilibrium of Type 1 where

\[ z_{0}^{CE} = \sum \pi_l z_{1s}^{CE} (q_{1s} + a_{1s} - \gamma) + \lambda_{1h}^{CE} \pi_h (1 - \gamma) \]

(34)

\[ z_{1s}^{CE} = \frac{A + 1 - \theta}{q_{1s} - \theta} \]

(35)

where \( \lambda_{2s}^{CE} > 0, \lambda_{1h}^{CE} = z_{0}^{CE} - z_{1h}^{CE} > 0 \) and \( \lambda_{1l}^{CE} = 0 \).

Before I prove Proposition 1, I prove Lemmas 1-2.

**Lemma 1** Conditional on Assumptions 1-6 and conditional on fire sale in the low state \( \frac{\partial B_l(k_{1l}^T)}{\partial k_{1l}^T} > 0 \) and \( \frac{\partial z_{1l}^{1,RP}}{\partial k_{1l}^T} > 0 \)

**Proof of Lemma 1.** From the first order condition of the policy maker with respect to \( B_l \)

\[
\frac{\tilde{F}''(k_{1l}^T) k_{1l}^T + \tilde{F}'(k_{1l}^T) - \theta}{\tilde{F}''(k_{1l}^T) k_{1l}^T + \tilde{F}'(k_{1l}^T) - \theta} = z_{1l}^{1,RP}(k_{1l}^T) = 1 + \delta'(B_l)
\]

From Assumption 4 and because \( 1 > \tilde{F}'(k_{1l}^T) \) if there is a fire sale

\[ z_{1l}^{1,RP} = \frac{\tilde{F}''(k_{1l}^T) k_{1l}^T + \tilde{F}'(k_{1l}^T) - \theta}{\tilde{F}''(k_{1l}^T) k_{1l}^T + \tilde{F}'(k_{1l}^T) - \theta} > 1 \]

Since \( z_{1l}^{1,RP} > 1 \) and \( \tilde{F}''(k_{1l}^T) < 0 \) and also from Assumption 4

\[
\frac{\partial z_{1l}^{1,RP}}{\partial k_{1l}^T} = \frac{\left( \tilde{F}''(k_{1l}^T) k_{1l}^T + \tilde{F}'(k_{1l}^T) \right) \left( 1 - z_{1l}^{1,RP} \right) - \tilde{F}''(k_{1l}^T) z_{1l}^{1,RP}}{\left[ \tilde{F}''(k_{1l}^T) k_{1l}^T + \tilde{F}'(k_{1l}^T) - \theta \right]} > 0
\]

Since \( \frac{\partial z_{1l}^{1,RP}}{\partial k_{1l}^T} > 0 \) and the deadweight loss function from the bail-out is convex, \( \delta''(B_l) > 0 \), one can show that larger fire sale leads to larger optimal bail-out

\[
\frac{\partial B_l(k_{1l}^T)}{\partial k_{1l}^T} = \frac{1}{\delta''(B_l)} \frac{\partial z_{1l}^{1,RP}}{\partial k_{1l}^T} > 0
\]
Lemma 2 Given Assumptions 1-5, considering a symmetric equilibrium and assuming no ex-ante regulation $\rho = 0$, there is never a fire sale in the high state, $q_{1h} = 1$. Given Assumption 2 and also given the additional Assumption 6, it is always the case that there is a fire sale in the low state, $q_{1l} < 1$.

Proof of Lemma 2. First I show that given a symmetric equilibrium, $q_{1h} = 1$. I can re-write the budget constraint in $t = 1$ and state $s$ as (taking into account that $d_{2s} = \theta k_{1s}$)

$$(k_{1s} - k_0)(q_{1s} - \theta) = (\theta + a_{1s} - \gamma)k_0 + B_s - d_{1s}$$

In the high state, from Assumption 5, $a_{1h} > 1 - \theta$. Also using the fact that the maximum promised payment in $t = 1$ in the high state is pinned down by the binding borrowing constraint, $d_{1h} = \theta(q_{1h} - \gamma)k_0$,

$$(k_{1s} - k_0)(q_{1s} - \theta) = (\theta + a_{1h} - \gamma)k_0 - d_{1h} \geq (\theta + a_{1h} - q_{1h} + (1 - \theta)(q_{1h} - \gamma))k_0 > 0$$

As a result, there is no fire sale in the high state, $q_{1h} = 1$, since $k_{1h} - k_0 > 0$.

The proof of $q_{1l} < 1$ proceeds in two steps. The first step is to show that, given Assumption 6, if there is no fire sale in the low state, the only possible equilibrium is the corner one where the bankers borrow to the maximum (Type 2 equilibrium). This implies that if the equilibrium is not the corner equilibrium of Type 2 then there must be a fire sale in the low state. Also since $k_0 = 0$ implies no fire sale in equilibrium (no borrowing in period zero and only lending to the maximum), proving the first step automatically implies that the corner solution $k_0 = 0$ is impossible as well justifying why I ignored the $k_0 \geq 0$ constraint when solving the Ramsey Problem. The second step is to show that given Assumption 6, even if the equilibrium of Type 2 is the optimal one there will be always a fire sale. Steps one and two are sufficient to prove that there is always a fire sale in equilibrium in the crisis state given the assumptions made. Finally, via simulations I prove that the set of parameters for which the Type 1 equilibrium is the optimal one is non-empty.

Step 1: I will prove that conditional on Assumption 2 being satisfied and if I assume that $q_{1l} = 1$ then $z_0^{CE} > z_{1s}^{CE}$ which implies that the only possible equilibrium if there are no fire sales is of Type 2. Re-write the first order condition with respect to $k_0$ as $\sum_s \pi_s [z_{1s}^{CE} (1 + a_{1s} - \gamma) + \lambda_{1s}(1 - \gamma)]$ $= z_0^{CE}$. If $\lambda_{1s} = 0$, then $z_{1s}^{CE} = z_0^{CE}$ which is impossible since Assumption 2 is violated because $\frac{A + 1 - \theta}{1 - \theta} \sum_s \pi_s (1 + a_{1s} - \gamma) = z_0^{CE} > z_{1s}^{CE} = \frac{A + 1 - \theta}{1 - \theta}$. Next consider the case $\lambda_{1l} = 0$ and $\lambda_{1h} > 0$. This case is impossible since it implies that $z_0^{CE} = z_{1l}^{CE} > z_{1h}^{CE} = z_0^{CE} - \lambda_{1h}$. However, from the first order condition with respect to $k_{1s}$, $z_{1s}^{CE} = z_{1h}^{CE}$ which is a contradiction. Similarly, the case $\lambda_{1h} = 0$ and $\lambda_{1l} > 0$ is impossible due to the same argument. Finally, I consider the case $\lambda_{1s} > 0$ which is the case where the bankers borrow to the maximum in $t = 0$ (type 2 equilibrium). From the first order condition with respect to $d_{1s}^0$, $z_{1s}^{CE} + \lambda_{1s}^{CE} = z_0^{CE} > z_{1s}^{CE}$. One can re-write $z_0^{CE}$ as

$$z_0^{CE} = \frac{A + 1 - \theta}{1 - \theta} \sum_s \pi_s \left[ (1 - \gamma)(1 - \theta) + a_{1s} \right] \left( 1 - \theta (1 - \gamma) \right)$$

(36)
\[ z_{CE}^{1s} = \frac{A + 1 - \theta}{1 - \theta} \] (37)

\[ z_{CE}^0 > z_{CE}^{1s} \text{ implies } z_{CE}^0 > \frac{(A + 1 - \theta)}{(1 - \theta)} \] (38)

Plugging equation 36 in 38, one can show that the condition \( z_{CE}^0 > z_{CE}^{1s} \) is satisfied as long as Assumption 2 is satisfied which completes the proof that as long as Assumption 2 is satisfied and there is no fire sale, the banker always optimally borrows to the maximum in \( t = 0 \) (the equilibrium is of Type 2).

Step 2: Next I prove that given Assumption 6 there is always a fire sale if, in equilibrium, the banker borrows to the maximum (the equilibrium is of Type 2). To do that I show that if the banker borrows to the maximum it is always the case that there exists and unique fire sale \( k_{11}^{T,\text{max}} \) such that \( k_{11}^{T,\text{max}} > 0 \).

I use the fact that in the high state there is no fire sale and \( q_{1h} = 1 \). To solve for \( \hat{b}_0^{\text{max}} \) as a function of \( k_{11}^{T,\text{max}} \) I use the budget constraint in \( t = 1 \) taking into account that all borrowing constraints are binding and the fact that \( \hat{b}_0^{\text{max}} = \sum_s \pi_s \theta (q_{1s} - \gamma) k_0 \). From \( n + \hat{b}_0^{\text{max}} = k_{0}^{\text{max}} \), one can solve for \( k_{0}^{\text{max}} \)

\[ k_{0}^{\text{max}} = \frac{n}{1 - \pi_h \theta (1 - \gamma) - \pi_t \theta \left( \tilde{F}' (k_{11}^{T}) - \gamma \right)} \] (39)

\[ \frac{\partial \hat{b}_0^{\text{max}} (k_{11}^{T})}{\partial k_{11}^{T}} = \frac{\pi_t \theta \tilde{F}'' (k_{11}^{T}) k_{0}^{\text{max}}}{1 - \pi_h \theta (1 - \gamma) - \pi_t \theta \left( \tilde{F}' (k_{11}^{T}) - \gamma \right)} < 0 \]

conditional on fire sale. From the period one budget constraint in the low state I can define the following function

\[ H \left( k_{11}^{T}; k_0 = k_0^{\text{max}} \right) = \left( \theta + a_{1l} - \gamma - \theta \left( \tilde{F}' (k_{11}^{T}) - \gamma \right) \right) k_{0}^{\text{max}} + B_l (k_{11}^{T}) + k_{11}^{T} \left( \tilde{F}' (k_{11}^{T}) - \theta \right) \] (40)

I show that \( H \left( k_{11}^{T,\text{max}} = k_0^{\text{max}}; k_0 = k_0^{\text{max}} \right) > 0 \) and given Assumption 6 \( H \left( k_{11}^{T} = 0; k_0 = k_0^{\text{max}} \right) < 0 \). Also \( \frac{\partial H (k_{11}^{T})}{\partial k_{11}^{T}} > 0 \). This guarantees that there exists and unique \( k_{11}^{T,\text{max}} > 0 \)

\[ H \left( k_{11}^{T,\text{max}} = k_0^{\text{max}}; k_0 = k_0^{\text{max}} \right) = \left( a_{1l} + (1 - \theta) \left( \tilde{F}' (k_{0}^{\text{max}}) - \gamma \right) \right) k_0^{\text{max}} + B_l (k_0^{\text{max}}) > 0 \]

\[ H \left( k_{11}^{T} = 0; k_0 = k_0^{\text{max}} \right) = \theta + a_{1l} - \gamma - \theta (1 - \gamma) \frac{n}{1 - \theta (1 - \gamma)} + (\delta')^{-1} \left( \frac{A}{1 - \theta} \right) < 0 \] (41)
where from Lemma 1 \( B_1^T(k_{11}^T) > 0 \). The fire sale that will emerge in equilibrium if the bank borrows to the maximum in \( t = 0 \) is pinned down by \( H(k_{11}^{T, \text{max}}) = 0 \). This completes the proof that given Assumptions 1-6 \( k_{11}^{T, \text{max}} > 0 \) is unique and exists and that there is always a fire sale in the low state.

**Proof of Proposition 1.**

First, I prove that the only two types of equilibria possible given a fire sale in the crisis state are of Type 1 and 2.

In order to characterize the equilibrium, I consider all four possible combinations of whether the Lagrangians on the period one borrowing constraint, \( \lambda_{1h} \) and \( \lambda_{1l} \), are greater than or equal to zero (given that I already proved that the period 2 borrowing constraints are always binding \( \lambda_{2s} > 0 \)).

If \( \lambda_{1h} = 0, \lambda_{1l} = 0 \) then \( z_{1h}^{CE} = z_{1l}^{CE} = z_0^{CE} \) and \( z_1^{CE} = \frac{A+1-\theta}{\theta q_{1l}} \). However since I proved that there is a fire sale only in the low state and no fire sale in the high state given the assumptions made then \( \frac{A+1-\theta}{\theta q_{1l}} = \frac{A+1-\theta}{\theta \theta \theta} \) is impossible and hence it will never be the case that \( \lambda_{1h} = 0, \lambda_{1l} = 0 \). If \( \lambda_{1h} = 0, \lambda_{1l} > 0 \) then \( z_{1h}^{CE} = z_{0}^{CE} \) and \( z_{1l}^{CE} - \lambda_{1l} = z_{1l}^{CE} < z_{1h}^{CE} \). I prove by contradiction that it is impossible that \( z_{1l}^{CE} < z_{1h}^{CE} \). If \( z_{1l}^{CE} < z_{1h}^{CE} \) implies that \( \frac{A+1-\theta}{\theta q_{1l}-\theta} < \frac{A+1-\theta}{\theta (1-\theta)} \) which is a contradiction since \( 1 > q_{1l} \).

**Type 1 equilibrium:** an interior equilibrium where \( \lambda_{1h} > 0, \lambda_{1l} = 0 \ (z_0^{CE} = z_{1l}^{CE} > z_{1h}^{CE}) \)

It will be the case that \( z_{1h}^{CE} = z_{1l}^{CE} + \lambda_{1h} \)

\[
\pi_l z_0^{CE} (q_{1l} + a_{1l} - \gamma) + \pi_h z_{1h}^{CE} (1 + a_{1h} - \gamma) + \lambda_{1h} \pi_h \theta (1 - \gamma) = z_0^{CE} \tag{42}
\]

Using the first order condition with respect to \( k_{1s} \)

\[
z_0^{CE} = z_{1l}^{CE} = \frac{A+1-\theta}{\theta q_{1l} - \theta} \quad z_{1h}^{CE} = \frac{A+1-\theta}{1-\theta}
\]

Notice that it is the case that \( z_{1l}^{CE} > z_{1h}^{CE} \) since \( q_{1l} < 1 \). From \( z_0^{CE} (k_{1l}^T) = z_{1l}^{CE} (k_{1l}^T) \) one can pin down the equilibrium \( k_{1l}^T \) if the equilibrium is of Type 1. Define period zero total borrowing of the consumer \( b_0 = k_0 - n = \sum \pi_s d_{1s} \). The rest of the endogenous variables are given by the following equations \( \{k_0, k_{1s}, d_{1s}, d_{2s}, B_t\} \)

\[
k_0 = \frac{\pi_l B_l + k_{1l}^T \pi_l (\tilde{F}'(k_{1l}^T) - \theta)}{(\pi_l (\gamma - \theta - a_{1l}) + 1 - \pi_h \theta (1 - \gamma))} + n
\]

\[
k_{1l} = k_0 - k_{1l}^T \tag{43}
\]

\[
k_{1h} = \frac{(1-\theta)(1-\gamma) + a_{1h} k_0}{(1-\theta)} \tag{44}
\]
Consider the case (44) exogenous variables. If the equilibrium \( k \) is unique.

\[ d_{2s} = \theta k_{1s}; \quad d_{1l} = \frac{1}{\pi_t} [(k_0 - n) - \pi_h \theta (1 - \gamma) k_0] \]  

(45)  

\[ d_{1h} = \theta (1 - \gamma) k_0 \]  

(46)  

and \( B_t = (\delta')^{-1} \left( z_{1l}^{1RP} (k_{1l}^T) - 1 \right) \). Finally \( p_{1s} = p_{2s} = 1 \) and \( q_{1h} = 1, q_{1l} = F' (k_{1l}^T) \).

**Type 2 equilibrium:** a corner equilibrium where \( \lambda_{1s} > 0 \) (\( z_0^{CE} > z_{1s}^{CE} \)).

Since I already proved that \( \lambda_{2s} > 0 \), in this type of equilibrium the banker borrows to the maximum in \( t = 0 \). If the equilibrium is the corner equilibrium of Type 2, the optimal \( k_{1l}^T \) is given by \( k_{1l}^{T,\text{max}} \) which is pinned down by equation 33. \( k_0 \) is given by equation 39, \( d_{2s}, d_{1l}, k_{1l}, k_{1h} \) and \( B_t \) are pinned down by the same equations as in the equilibrium of Type 1. \( d_{1l} = \theta (q_{1l} - \gamma) k_0 \).

Next I prove existence and uniqueness of the equilibrium.

One can solve for the equilibrium by solving for \( k_0 \). First, I show that for every \( k_0 \in [0, k_0^{\text{max}}] \) there exists an unique \( k_{1l}^T \). I will consider two regions for \( k_0 \) separately. If the equilibrium \( k_0 \) is such that \( k_0 \in [0, \hat{k}_0] \) then there will be no fire sale, \( k_{1l}^T = 0 \), where I will derive \( \hat{k}_0 \) as a function of exogenous variables. If the equilibrium \( k_0 \) is such that \( k_0 \in (\hat{k}_0, k_0^{\text{max}}] \) then there will be a fire sale \( k_{1l}^T > 0 \) and \( k_{1l}^T \) is unique.

Since I proved that the only possible case is \( \lambda_{1l} > 0, \lambda_{2s} > 0 \), from the period zero budget constraint and equation 44

\[ d_{1l} (k_{1l}^T; k_0) = \min \left\{ \frac{1}{\pi_t} [k_0 [1 - \pi_h \theta (1 - \gamma)] - n], \theta (q_{1l} - \gamma) k_0 \right\} \]

From the budget constraint in the low state in \( t = 1 \), define

\[ H (k_{1l}^T; k_0) = (\theta + a_{1l} - \gamma) k_0 + k_{1l}^T \left( \tilde{F}' (k_{1l}^T) - \theta \right) + B_t (k_{1l}^T) - d_{1l} (k_{1l}^T; k_0) \]

where \( B_t (k_{1l}^T) \) implies that the bail-out is a function of the fire sale in the low state. First, consider how the function \( H (k_{1l}^T; k_0) \) behaves in the range \( k_{1l}^T \in [0, k_0] \). First I show that \( H (k_{1l}^T = k_0; k_0) > 0 \) for every \( k_0 \)

\[ H (k_{1l}^T = k_0; k_0) = \left( \theta + a_{1l} - \gamma + \left( \tilde{F}' (k_0) - \theta \right) \right) k_0 + B_t (k_0) - d_{1l} (k_0; k_0) \]

\[ \geq \left( a_{1l} + (1 - \theta) \left( \tilde{F}' (k_0) - \gamma \right) \right) k_0 + B_t (k_0) > 0 \]

where for the first inequality I used the fact that \( d_{1l} (k_{1l}^T; k_0) \leq \theta (q_{1l} - \gamma) k_0 \). Next I show that \( H (k_{1l}^T = 0; k_0) > 0 \) if \( k_0 \in [0, \hat{k}_0] \) and \( H (k_{1l}^T = 0; k_0) < 0 \) if \( k_0 \in (\hat{k}_0, k_0^{\text{max}}] \).

I already showed that if \( k_0 = k_0^{\text{max}}, H (k_{1l}^T = 0; k_0 = k_0^{\text{max}}) < 0 \) (inequality 41) Therefore, consider the case

\[ H (k_{1l}^T = 0; k_0) = (\theta + a_{1l} - \gamma) k_0 + B_t (0) - \frac{1}{\pi_t} [k_0 [1 - \pi_h \theta (1 - \gamma)] - n] \]

\[ = \frac{1}{\pi_t} [n + k_0 \{ \pi_l (\theta + a_{1l} - \gamma) - 1 + \pi_h \theta (1 - \gamma) \}] + (\delta')^{-1} \left( \frac{A}{1 - \theta} \right) \]
Since \( \pi_l (\theta + a_{1l} - \gamma) - 1 + \pi_h \theta (1 - \gamma) < 0 \), the higher \( k_0 \) is the more negative \( H (k_{1l}^T = 0; k_0) \) is. One can show that

\[
H (k_{1l}^T = 0; k_0) \begin{cases} 
> 0 & \text{if } k_0 \leq \hat{k}_0 \\
< 0 & \text{if } k_0 > \hat{k}_0 
\end{cases}
\]

where

\[
\hat{k}_0 = \frac{\pi_l (\delta')^{-1} \left( \frac{A}{1 - \theta} \right) + n}{\{1 + \pi_l (\gamma - \theta - a_{1l}) - \pi_h \theta (1 - \gamma)\}}
\]

I prove that \( H (k_{1l}^T; k_0) \) is continuous in \( k_{1l}^T \) and \( \frac{\partial H (k_{1l}^T; k_0)}{\partial k_{1l}^T} > 0 \).

The continuity of \( H (k_{1l}^T; k_0) \) follows from \( B_l (k_{1l}^T) \) and \( \tilde{F}^l (k_{1l}^T) \) being continuous with respect to \( k_{1l}^T \). Since in the region \([0, \hat{k}_0] \), \( H (k_{1l}^T = 0; k_0) > 0 \) and \( H (k_{1l}^T = k_0; k_0) > 0 \) and \( \frac{\partial H (k_{1l}^T; k_0)}{\partial k_{1l}^T} > 0 \), it follows that \( k_{1l}^T (k_0) = 0 \) if \( k_0 \in [0, \hat{k}_0] \). In the region \((\hat{k}_0, k_0^{\max}) \), \( H (k_{1l}^T = 0; k_0) > 0 \) and \( H (k_{1l}^T = k_0; k_0) < 0 \) and \( \frac{\partial H (k_{1l}^T; k_0)}{\partial k_{1l}^T} > 0 \) and \( H (k_{1l}^T; k_0) \) is continuous. As a result, there exists an unique \( k_{1l}^T (k_0) > 0 \) if \( k_0 \in (\hat{k}_0, k_0^{\max}) \). This completes the proof that for every \( k_0 \in [0, k_0^{\max}] \) there exists an unique \( k_{1l}^T \geq 0 \).

I can totally differentiate \( H (k_0) = 0 \) with respect to \( k_0 \) to solve for \( \frac{\partial k_{1l}^T (k_0)}{\partial k_0} \) in the relevant range \( k_0 \in (\hat{k}_0, k_0^{\max}) \) where there is a fire sale. In that range, for a given \( k_0 \), \( k_{1l}^T \) is pinned down by setting \( H (k_{1l}^T; k_0) = 0 \).

Totally differentiate \( H (k_{1l}^T; k_0) = 0 \) with respect to \( b_0 \). From Lemma 1 and also from Assumption 4

\[
\frac{\partial k_{1l}^T (k_0)}{\partial k_0} = \begin{cases} 
\frac{1}{\pi_l [1 - \pi_h \theta (1 - \gamma)] + \pi_h A + (1 - \theta) [a_{1l} + (1 - \theta)(1 - \gamma)] + \pi_l (A + 1 - \theta)}{B_l (k_{1l}^T) + \tilde{F}^l (k_{1l}^T) - \theta + \pi_l (A + 1 - \theta)} > 0 & \text{if } (k_0, k_0^{\max}) \\
0 & \text{if } (0, \hat{k}_0] 
\end{cases}
\]

Consider the interior equilibrium of Type 1 where \( \lambda_{th} > 0, \lambda_{lh} > 0 \) and \( \lambda_{1l} = 0 \). Define the following function which will be used to pin down the equilibrium \( k_0 \)

\[
\psi^{CE} (k_0) = z_{1l}^{CE} (k_0) - z_0^{CE} (k_0) \text{ if } k_0 \in (\hat{k}_0, k_0^{\max})
\]

where

\[
z_0^{CE} (k_0) = -\frac{\pi_l z_{1l}^{CE} (k_0) (\gamma - \theta - a_{1l}) + \pi_h A + \frac{1}{1 - \theta} [a_{1h} + (1 - \theta) (1 - \gamma)] + \pi_l (A + 1 - \theta)}{1 - \pi_h \theta (1 - \gamma)}
\]

\[
z_{1l}^{CE} (k_0) = \frac{A + 1 - \theta}{\tilde{F}^l (k_{1l}^T (k_0)) - \theta}
\]

Given the assumptions made, I proved in Lemma 2, that in equilibrium there will be always a fire sale, I consider only the relevant range for \( k_0 \), \( k_0 \in (\hat{k}_0, k_0^{\max}) \) in which there will be a fire
sale. If $\psi^{CE}(k_0) = 0$ then the equilibrium is interior and of type 1. If for every $k_0$ in the range $(k_0, k_0^{max})$, $\psi^{CE}(k_0) < 0$ then the equilibrium is of Type 2. Also I will prove that if the equilibrium is of Type 2 it cannot be of type 1. In other words, I will prove that $z^{CE} < z^{CE}_1 < 0$ where $\lambda_{1l} = 0, \lambda_{1h} > 0, \lambda_{2s} > 0$ iff $z^{CE}_0 < z^{CE}_{1s} > 0$ where $\lambda_{ts} > 0$. Finally, I will show that given the assumptions made, it will be never the case that $\psi^{CE}(k_0) > 0$ for all $k_0 \in [k_0, k_0^{max}]$.

Since $-\frac{\partial k_{1l}^T(k_0)}{\partial k_0} > 0$

$$\frac{\partial z^{CE}_{1l}(k_0)}{\partial k_0} = - \frac{\partial k_{1l}^T(k_0)}{\partial k_0} \tilde{F}''(k_{1l}^T(k_0)) \frac{z^{CE}_{1l}(k_0)}{(\tilde{F}'(k_{1l}^T(k_0)) - \theta)} > 0$$

$$\frac{\partial z^{CE}_0(k_0)}{\partial k_0} = - \frac{\pi_l (\gamma - \theta - a_{1l})}{[1 - \pi_h\theta(1 - \gamma)]} \frac{\partial z^{CE}_{1l}(k_0)}{\partial k_0} < 0$$

$$\frac{\partial \psi^{CE}(k_0)}{\partial k_0} = \frac{\partial z^{CE}_{1l}(k_0)}{\partial k_0} - \frac{\partial z^{CE}_0(k_0)}{\partial k_0} = \left(1 + \frac{\pi_l (\gamma - \theta - a_{1l})}{[1 - \pi_h\theta(1 - \gamma)]}\right) \frac{\partial z^{CE}_0(k_0)}{\partial k_0} > 0$$

Next I show that given the assumptions made, it will be never the case that $\psi^{CE}(k_0) > 0$ for all $k_0 \in [k_0, k_0^{max}]$. I already proved that $\frac{\partial \psi^{CE}(k_0)}{\partial k_0} > 0$. As a result, it’s sufficient to prove that $\psi^{CE}(k_0 = \hat{k}_0) < 0$. Since by definition if $k_0=\hat{k}_0$, there will be no fire sale in the low state, I can re-write $\psi^{CE}(k_0 = \hat{k}_0)$ as

$$\psi^{CE}(k_0 = \hat{k}_0) = - \frac{A + 1 - \theta \sum \pi_s (a_{1s} + 1 - \gamma) - 1}{1 - \theta} \frac{1 - \pi_h\theta(1 - \gamma)}{1 - \pi_h\theta(1 - \gamma)} < 0 \quad (48)$$

The result follows from the Assumption 2. Next I prove that $z^{CE}_{1l} - z^{CE}_0 < 0$ where $\lambda_{1l} = 0, \lambda_{1h} > 0, \lambda_{2s} > 0$ iff $z^{CE}_{1l} - z^{CE}_{1s} > 0$ where $\lambda_{ts} > 0$. First, I show that if $z^{CE}_{1l} - z^{CE}_0 < 0$ where $\lambda_{1l} = 0, \lambda_{1h} > 0, \lambda_{2s} > 0$, then $z^{CE}_{1l} - z^{CE}_{1s} > 0$ where $\lambda_{ts} > 0$ which will be sufficient to finish the prove that if the equilibrium is not of Type 1 it is of Type 2. If $\lambda_{1l} = 0, \lambda_{1h} > 0, \lambda_{2s} > 0$

$$z^{CE}_{1l} - z^{CE}_0 = \frac{A + 1 - \theta}{(q_{1l} - \theta)} \left[1 - \pi_h\theta(1 - \gamma) + \pi_l (\gamma - \theta - a_{1l})\right] - \pi_h \frac{A + 1 - \theta}{1 - \theta} \left[a_{1h} + (1 - \theta)(1 - \gamma)\right] - \pi_l (A + 1 - \theta) \frac{1}{1 - \pi_h\theta(1 - \gamma)} < 0$$

which implies

$$\frac{1}{(q_{1l} - \theta)} \left[1 - \pi_h\theta(1 - \gamma) + \pi_l (\gamma - a_{1l} - q_{1l})\right] - \pi_h \frac{1}{1 - \theta} \left[a_{1h} + (1 - \theta)(1 - \gamma)\right] < 0 \quad (49)$$

Next let’s consider the case where all the borrowing constraints are binding, $\lambda_{ts} > 0$. Notice that the equations for $z^{CE}_{1l}$ and $z^{CE}_0$ will change from the case where $\lambda_{1l} = 0$. From the first order condition with respect to $k_{1s}$ it follows that $\frac{A + 1 - \theta}{q_{1s} - \theta} = z^{CE}_{1l} > z^{CE}_{1h} = \frac{A + 1 - \theta}{1 - \theta}$. Therefore, it is sufficient to show that given Assumption 7, it is always the case that $z^{CE}_{1l} - z^{CE}_0 < 0$ if equation 49 is satisfied.

$$z^{CE}_0(k_0) = \frac{\sum \pi_s \frac{A + 1 - \theta}{q_{1s} - \theta} ((q_{1s} - \gamma)(1 - \theta) + a_{1s})}{1 - \sum \pi_s \theta(q_{1s} - \gamma)}$$

55
which implies that even if the equilibrium is of Type 2 there will be a resale in the crisis state. I
Type 2 to never occur. Consider the case where
unique
Given Assumptions 1-6, considering a symmetric equilibrium for the Central Plan-
as long as

Finally, I prove that Assumption 7 combined with Assumption 6, which guarantees that there is always a fire sale in the crisis state, are necessary and sufficient conditions for equilibrium of Type 2 to never occur. Consider the case where \( \lambda_{ts} > 0 \) and assume that Assumption 6 is satisfied which implies that even if the equilibrium is of Type 2 there will be a fire sale in the crisis state. I already showed that \( z_{1}^{CE} > z_{1}^{0} \). Therefore, it is sufficient to show that given Assumption 7, it is always the case that \( z_{1}^{CE} > z_{0}^{CE} \) and, as a result, equilibrium of Type 2 will never occur. Since I already solved for \( z_{1}^{CE} (k_{0}) - z_{0}^{CE} (k_{0}) \) in equation 50 one can show that \( z_{1}^{CE} (k_{0}) - z_{0}^{CE} (k_{0}) > 0 \) as long as

\[
1 + \pi_{l} (\gamma - a_{1l} - F' (k_{1l}^{T})) > \pi_{h} \left\{ \frac{(F' (k_{1l}^{T}) - \theta)}{(1 - \theta)} a_{1h} + F' (k_{1l}^{T}) (1 - \gamma) \right\}
\]

where \( k_{1l}^{T} \) is pinned down by \( H \left( k_{1l}^{T, max}; b_{0} = \hat{b}_{0}^{max} \right) = 0 \).

### 10.5.2 Proposition 2

**Proposition 2** Given Assumptions 1-6, considering a symmetric equilibrium for the Central Planner’s problem with no commitment, there is never a fire sale in the high state, \( q_{1h} = 1 \) and there is a fire sale in the low state, \( q_{1l} < 1 \). The equilibrium exists and is unique and is one of the following types. 1) \( z_{1}^{CP} > z_{1}^{CP} \) (interior equilibrium) and 2) \( z_{1}^{CP} > z_{1}^{CP} \) (corner equilibrium where the banker borrows to the maximum in \( t = 0 \)). The optimal bail-out is determined by

\[
1 + \delta (B_{t}) = z_{1}^{1, CP} = \frac{F'' (k_{1l}^{T}) k_{1l}^{T} + A + 1 - \theta}{F'' (k_{1l}^{T}) k_{1l}^{T} + F' (k_{1l}^{T}) - \theta}
\]

If also the following assumption

\[
\frac{F'' (k_{1l}^{T}) k_{1l}^{T} + A + 1 - \theta}{F' (k_{1l}^{T}) - \theta + F'' (k_{1l}^{T}) k_{1l}^{T}} (1 - \pi_{h} \theta [1 - \gamma] + \pi_{l} (\gamma - \theta - a_{1l}))
\]

Assumption 7’

\[
\frac{A + 1 - \theta}{(1 - \theta)} [\pi_{l} (1 - \theta) + \pi_{h} ((1 - \theta) (1 - \gamma) + a_{1h})] > 0
\]
is satisfied where \( k_{1l}^T \) is pinned down by equation 33, the only possible equilibrium is the interior equilibrium of Type 1 where

\[
\sum \pi_s \left( -F'' \left( k_{1s}^T \right) k_{1s}^T + z_{1s}^{CP} \left( F' \left( k_{1s}^T \right) + a_{1s} - \gamma + F'' \left( k_{1s}^T \right) k_{1s}^T \right) \right) + \pi_h \lambda_{1h}^{CP} \theta [1 - \gamma] = z_0^{CP}
\]

\[
z_{1h}^{CP} = \frac{A + 1 - \theta}{1 - \theta}
\]

\[
z_{1l}^{CP} = \frac{F'' \left( k_{1l}^T \right) k_{1l}^T + A + 1 - \theta}{F' \left( k_{1l}^T \right) - \theta + F'' \left( k_{1l}^T \right) k_{1l}^T}
\]

\[
\lambda_{2s}^{CP} > 0, \lambda_{1h}^{CP} > 0 \text{ and } \lambda_{1l}^{CP} = 0. \text{ Also } z_{1s}^{CP} = z_{1s}^{0,CP}.
\]

Before I prove Proposition 2 I prove Lemmas 3 and 4.

**Lemma 3** Conditional on Assumptions 1-6 and conditional on a fire sale in the low state, \( \frac{\partial B_s(k_{1l}^T)}{\partial k_{1l}} > 0 \) and \( \frac{\partial z_{1l}^{1,CP}}{\partial k_{1l}} > 0 \).

**Proof of Lemma 3.** The proof is identical to the proof of Lemma 1 since \( z_{1l}^{1,CP} \left( k_{1l}^T \right) = z_{1l}^{1,RP} \left( k_{1l}^T \right) \).

**Lemma 4** Given Assumption 5, considering a symmetric equilibrium, there is never a fire sale in the high state, \( q_{1h} = 1 \). Given Assumption 2 and also given the additional Assumption 6, it is always the case that there is a fire sale in the low state, \( q_{1l} < 1 \).

**Proof of Lemma 4.** Given that one can re-write the first order conditions of the Central Planner’s problem, assuming that \( q_{1l} = 1 \), as

\[
\sum \pi_s \left( -\frac{\partial B_s}{\partial k_0} \left( 1 + \delta' \left( B_s \right) \right) + z_{1s}^{CP} \left( 1 + a_{1s} - \gamma + \frac{\partial B_s}{\partial k_0} \right) + \lambda_{1s}^{CP} \theta [1 - \gamma] \right) = z_0^{CP}
\]

since \( z_{1l}^{1,CP} \left( k_{1l}^T \right) = \frac{A + 1 - \theta}{1 - \theta} \) implies that \( \frac{\partial z_{1l}^{1,CP}}{\partial k_{1l}} = 0 \), \( \frac{\partial B_s}{\partial k_0} = 1 \), \( \frac{\partial B_s}{\delta' \left( B_s \right) \delta k_{1s}} = 0 \) and \( \frac{\partial B_s}{\delta k_{1s}} = -\frac{1}{\delta' \left( B_s \right)} \frac{\partial z_{1l}^{1,CP}}{\partial k_{1s}} = 0 \). As a result,

\[
\sum \pi_s \left( z_{1s}^{CP} \left( 1 + a_{1s} - \gamma \right) + \lambda_{1s}^{CP} \theta [1 - \gamma] \right) = z_0^{CP}
\]

where \( \frac{A + 1 - \theta}{1 - \theta} = z_{1s}^{CP} \) and \( z_0^{CP} - z_{1s}^{CP} = \lambda_{1s}^{CP} \). The equations above coincide with equations in the proof of Lemma 2. The rest of the proof is identical to the proof in Lemma 2.

**Proof of Proposition 2.**

First I prove that the only two types of equilibria possible are of Type 1 and 2 similarly to the Ramsey Problem with no minimum capital requirement.

In order to characterize the equilibrium, I consider all four possible combinations of whether \( \lambda_{1h} \) and \( \lambda_{1l} \) are greater than or equal to zero.
If $\lambda_{1h} = 0, \lambda_{1l} = 0$ then $z_{1h}^{CP} = z_{1l}^{CP} = z_{0}^{CP}$. Plugging equation 21 in equation 24a one gets $z_{1s}^{CP} (k_{1s}^{T}) = \frac{[F''(k_{1s}^{T}) k_{1s}^{T} + A + 1 - \theta]}{F''(k_{1s}^{T}) k_{1s}^{T} + F'(k_{1s}^{T}) - \theta}$. However, since there is a fire sale only in the low state and no fire sale in the high state given the assumptions made, then one can prove that $z_{1l}^{CP} = \frac{[F''(k_{1s}^{T}) k_{1s}^{T} + A + 1 - \theta]}{F''(k_{1s}^{T}) k_{1s}^{T} + F'(k_{1s}^{T}) - \theta} > A + 1 - \theta = z_{1h}^{CP}$. This is true since $(1 - \tilde{F}'(k_{1s}^{T}))(A + 1 - \theta) > \tilde{F}''(k_{1s}^{T}) k_{1s}^{T} A$.

Hence it will never be the case that $\lambda_{1h} = 0, \lambda_{1l} = 0$.

If $\lambda_{1h} = 0, \lambda_{1l} > 0$, then $z_{1l}^{CP} + A = \frac{A + 1 - \theta}{1 - \theta}$ and $z_{1l}^{CP} - A = \frac{A + 1 - \theta}{1 - \theta}$.

I prove by contradiction that it is impossible that $z_{1l}^{CP} < z_{1l}^{CP}$. After plugging in $z_{1l}^{CP}$ given by equation 21 and taking into account that $z_{1l}^{CP} = z_{0}^{CP}$, one can re-write inequality 53 as $\frac{\tilde{F}''(k_{1l}^{T}) k_{1l}^{T} + A + 1 - \theta}{\tilde{F}''(k_{1l}^{T}) k_{1l}^{T} + \tilde{F}'(k_{1s}^{T}) - \theta} < z_{1l}^{CP} = \frac{A + 1 - \theta}{1 - \theta}$, which is a contradiction. As a result, it is impossible that $\lambda_{1h} = 0, \lambda_{1l} > 0$.

**Type 1 equilibrium:** $\lambda_{1h} > 0, \lambda_{1l} = 0$ ($z_{0}^{CP} = z_{1l}^{CP} > z_{1h}^{CP}$)

Notice that $z_{1l}^{CP} = z_{1l}^{CP} + \lambda_{1h}^{CP} > z_{1h}^{CP}$ and from the first order condition with respect to $k_{1h}, z_{1h}^{CP} = \frac{A + 1 - \theta}{1 - \theta}$

$$z_{0}^{CP} = \frac{\pi_{l} \left( -\tilde{F}''(k_{1l}^{T}) k_{1l}^{T} + z_{1l}^{CP} \left( \tilde{F}'(k_{1l}^{T}) + A_{1l} - \gamma + \tilde{F}''(k_{1l}^{T}) k_{1l}^{T} \right) + \pi_{h} z_{1h}^{CP} \left( (1 - \theta) (1 - \gamma) + a_{1h} \right) \right)}{\left( 1 - \pi_{h} \theta \right) \left( 1 - \gamma \right)}$$ (54)

Plugging in for $z_{1l}^{CP}$, from the first order condition with respect to $k_{1l}$

$$z_{1l}^{CP} = z_{1l}^{CP} = \frac{\tilde{F}''(k_{1l}^{T}) k_{1l}^{T} + A + 1 - \theta}{\tilde{F}''(k_{1l}^{T}) k_{1l}^{T} + \tilde{F}'(k_{1s}^{T}) - \theta + \tilde{F}''(k_{1l}^{T}) k_{1l}^{T}}$$ (55)

The rest of the variables are determined by the same set of equations as the ones in the Type 1 equilibrium in Proposition 1. The equilibrium of Type 1 is possible since we already proved that $z_{1l}^{CP} > z_{1l}^{CP}$.

**Type 2 equilibrium:** If $\lambda_{1h} > 0, \lambda_{1l} > 0$ ($z_{0}^{CP} < z_{1l}^{CP} < z_{1h}^{CP}$)

Since I already proved that $\lambda_{2s} > 0$, in this type of equilibrium the banker borrows to the maximum in $t = 0$. From the first order condition with respect to $k_{1s}, z_{0}^{CP} = z_{1s}^{CP} + \lambda_{1s} > z_{1s}^{CP}$

$$z_{0}^{CP} = \frac{\pi_{l} \left[ \tilde{F}''(k_{1l}^{T}) k_{1l}^{T} + \frac{\partial B_{il}}{\partial T_{il}} + \frac{1 - \theta}{1 - \theta} \theta + a_{1l} - \gamma + \theta \left( \tilde{F}'(k_{1s}^{T}) - \gamma \right) \right]}{\tilde{F}''(k_{1l}^{T}) k_{1l}^{T} + \frac{\partial B_{il}}{\partial T_{il}} + \tilde{F}'(k_{1s}^{T}) - \theta + \tilde{F}''(k_{1l}^{T}) k_{1l}^{T}} + \theta \frac{A + 1 - \theta}{1 - \theta} \left[ \pi_{l} \left( 1 - \theta \right) + \pi_{h} \left( (1 - \theta) (1 - \gamma) + a_{1h} \right) \right]$$ (56)
and \( z_{1l}^{CP} \) is given by equation 53 and \( z_{0l}^{CP} = \frac{A+1-\theta}{1-\sigma} \). The rest of the variables are determined by the same set of equations as the ones in the Type 2 equilibrium in Proposition 1.

**Next I prove existence and uniqueness.**

As in the proof of Proposition 1, one can show that for every \( k_0 \in [0,k_0^{max}] \) there exists an unique \( k_{1l}^{T} \) and if the equilibrium \( k_0 \) is such that \( k_0 \in (\hat{k}_0, k_0^{max}] \) then there will be a fire sale, \( k_{1l}^{T} > 0 \), where \( \hat{k}_0 \) is pinned down by equation 47. Also as in Proposition 1, one can prove that \( \frac{\partial k_{1l}^{T}(k_0)}{\partial k_0} > 0 \) if \( k_0 \in (\hat{k}_0, k_0^{max}] \).

I take into account that it will be always the case that \( \lambda_{th}^{CP} > 0, \lambda_{2l}^{CP} > 0 \). Consider the only interior equilibrium of Type 1 which implies \( \lambda_{l}^{CP} = 0 \). Following the same steps as in the Proof of Proposition 1, define the following function which will be used to pin down the equilibrium \( k_0 \)

\[
\psi^{CP}(k_0) = z_{1l}^{CP}(k_0) - z_{0l}^{CP}(k_0) \quad \text{if} \quad k_0 \in [\hat{k}_0, k_0^{max}]
\]

where

\[
z_{0l}^{CP}(k_0) = \frac{-\pi_{l}z_{1l}^{CP}(k_0)(\gamma - \theta - a_{l1}) + A + 1 - \theta}{(1 - \pi_{h}\theta [1 - \gamma])}
\]

\[
z_{1l}^{CP}(k_0) = \frac{\bar{F}''(k_{1l}^{T}(k_0))k_{1l}^{T}(k_0) + A + 1 - \theta}{\bar{F}'(k_{1l}^{T}(k_0)) - \theta + \bar{F}''(k_{1l}^{T}(k_0))k_{1l}^{T}(k_0)}
\]

If \( \psi^{CP}(k_0) = 0 \), then the equilibrium is interior and of Type 1. If for every \( k_0 \) in the range \([\hat{k}_0, k_0^{max}]\), \( \psi^{CP}(k_0) < 0 \) then the equilibrium is a corner equilibrium where it is optimal to borrow to the maximum in period zero against the high and the low states. I will show that given the assumptions made, it will be never the case that \( \psi^{CP}(k_0) > 0 \) for all \( k_0 \in [\hat{k}_0, k_0^{max}] \). Finally, I will prove that \( z_{1l}^{CE} - z_{0l}^{CE} < 0 \) where \( \lambda_{l1} = 0, \lambda_{l1h} > 0, \lambda_{2s} > 0 \) iff \( z_{0l}^{CE} - z_{1l}^{CE} > 0 \) where \( \lambda_{ts} > 0 \). Since \( \frac{\partial k_{1l}^{T}(k_0)}{\partial k_0} > 0 \) and since \( z_{1l}^{CE} = z_{1l}^{CP} \) in the interior equilibrium.

\[
\frac{\partial z_{1l}^{CP}(k_0)}{\partial k_0} = \frac{\partial k_{1l}^{T}(k_0)}{\partial k_0} z_{1l}^{CP} > 0
\]

\[
\frac{\partial z_{0l}^{CE}(k_0)}{\partial k_0} = -\frac{\pi_{l}(\gamma - \theta - a_{l1})}{(1 - \pi_{h}\theta [1 - \gamma])} \frac{\partial z_{1l}^{CP}(k_0)}{\partial k_0} < 0
\]

\[
\frac{\partial \psi^{CP}(k_0)}{\partial k_0} = \frac{\partial z_{1l}^{CP}(k_0)}{\partial k_0} - \frac{\partial z_{0l}^{CE}(k_0)}{\partial k_0} = \left( 1 + \frac{\pi_{l}(\gamma - \theta - a_{l1})}{1 - \pi_{h}\theta [1 - \gamma]} \right) \frac{\partial z_{1l}^{CP}(k_0)}{\partial k_0} > 0
\]

Next I show that given the assumptions made, it will be never the case that \( \psi^{CE}(k_0) > 0 \) for all \( k_0 \in [\hat{k}_0, k_0^{max}] \). I already proved that \( \frac{\partial \psi^{CE}(k_0)}{\partial k_0} > 0 \). As a result, it’s sufficient to prove that \( \psi^{CE}(k_0 = \hat{k}_0) < 0 \). Since by definition if \( k_0 = \hat{k}_0 \), there will be no fire sale in the low state, \( \psi^{CP}(k_0 = \hat{k}_0) \) is the same as \( \psi^{CE}(k_0 = \hat{k}_0) \) given by equation 48. Next I prove that \( z_{1l}^{CP} - z_{0l}^{CP} < 0 \)
where \( \lambda_{ll} = 0, \lambda_{lh} > 0, \lambda_{ls} > 0 \) iff \( z_0^{CP} - z_1^{CP} > 0 \) where \( \lambda_{ts} > 0 \). First, I show that if \( z_0^{CP} - z_1^{CP} < 0 \) where \( \lambda_{ll} = 0, \lambda_{lh} > 0, \lambda_{ls} > 0 \), then \( z_0^{CP} - z_1^{CP} > 0 \) where \( \lambda_{ts} > 0 \) which will be sufficient to finish the proof that if the equilibrium is not of Type 1 it is of Type 2. If \( \lambda_{ll} = 0, \lambda_{lh} > 0, \lambda_{ls} > 0 \)

\[
z_1^{CP} - z_0^{CP} = \frac{1}{(1 - \pi_h \theta [1 - \gamma])} \left\{ \frac{\tilde{F}''(k_{11}^T)k_{11}^T + A + 1 - \theta}{F'(k_{11}^T) - \theta + F''(k_{11}^T)k_{11}^T} (1 - \pi_h \theta [1 - \gamma] + \pi_l (\gamma - \theta - a_{1l})) - \frac{A + 1 - \theta}{(1 - \theta)} [\pi_l (1 - \theta) + \pi_h ((1 - \theta)(1 - \gamma) + a_{1h})] \right\} < 0
\]

which implies

\[
\frac{\tilde{F}''(k_{11}^T)k_{11}^T + A + 1 - \theta}{F'(k_{11}^T) - \theta + F''(k_{11}^T)k_{11}^T} (1 - \pi_h \theta [1 - \gamma] + \pi_l (\gamma - \theta - a_{1l})) - \frac{A + 1 - \theta}{(1 - \theta)} [\pi_l (1 - \theta) + \pi_h ((1 - \theta)(1 - \gamma) + a_{1h})] < 0
\]

(60)

Next let’s consider the case where \( \lambda_{ts} > 0 \) and I will prove that if the inequality 60 is satisfied then \( z_1^{CP} - z_0^{CP} < 0 \). One can focus only on \( z_1^{CP} \), since \( z_1^{CP} > z_{1h}^{CP} > z_0^{CP} \) and \( z_1^{CP} \) are given by equations 56 and 53. Re-write \( z_1^{CP} - z_0^{CP} < 0 \) and simplify, one gets exactly the same inequality given by 60. Given that the inequalities are identically, it is obvious that it will be also the case that if \( z_0^{CP} - z_1^{CP} > 0 \) where \( \lambda_{ts} > 0 \) then \( z_1^{CP} - z_0^{CP} < 0 \) where \( \lambda_{ll} = 0, \lambda_{lh} > 0, \lambda_{ls} > 0 \).

As a result, the only possible equilibria are either borrow to the maximum against both states in period zero \( (z_0^{CE} > z_{1l}^{CE} > z_{1h}^{CE}) \) or the interior equilibrium \( z_0^{CE} = z_{1l}^{CE} > z_{1h}^{CE} \). Given that \( \psi' (k_0) > 0 \) in the relevant range \( k_0 \in [k_0, k_0^{max}] \) and given that I proved that \( z_{1l}^{CE} - z_0^{CE} < 0 \) where \( \lambda_{ll} = 0, \lambda_{lh} > 0, \lambda_{ls} > 0 \) iff \( z_0^{CE} - z_{1s}^{CE} > 0 \) where \( \lambda_{ts} > 0 \), the interior equilibrium exists and is unique.

Finally, I prove that Assumption 7’ combined with Assumption 6, which guarantees that there is always a fire sale in the crisis state, are necessary and sufficient conditions for equilibrium of Type 2 to never occur. Consider the case where \( \lambda_{ts} > 0 \) and assume that Assumption 6 is satisfied which implies that even if the equilibrium is of Type 2 there will be a fire sale in the crisis state. It is sufficient to show that given Assumption 7’, it is always the case that \( z_0^{CP} - z_0^{CP} > 0 \) and, as a result, equilibrium of Type 2 will never occur. One can show that \( z_1^{CP} - z_0^{CP} > 0 \) as long as

\[
\frac{\tilde{F}''(k_{11}^T)k_{11}^T + A + 1 - \theta}{F'(k_{11}^T) - \theta + F''(k_{11}^T)k_{11}^T} (1 - \pi_h \theta [1 - \gamma] + \pi_l (\gamma - \theta - a_{1l})) - \frac{A + 1 - \theta}{(1 - \theta)} [\pi_l (1 - \theta) + \pi_h ((1 - \theta)(1 - \gamma) + a_{1h})] > 0
\]

where \( k_{11}^T \) is pinned down by equation 33.

10.5.3 Overinvestment


**Proposition 3** Conditional on assumptions 1-6 and comparing the constrained Central Planner’s allocation without commitment and the Competitive Equilibrium from the Ramsey Problem without commitment where \( \rho = 0 \), there is always overinvestment, \( k^C_P < k^{CE}_0 \), if the equilibrium is of Type 1 for the Central Planner (interior equilibrium) and there is no overinvestment, \( k^C_P = k^{CE}_0 \), if the equilibrium is of Type 2 for the Central Planner (corner equilibrium).

**Proof of Proposition 3.** Since \( \psi^{CE'} (k_0) > 0 \) and \( \psi^{CE'} (k_0) > 0 \), it is sufficient to prove that \( \psi^{CP} (k_0) > \psi^{CE} (k_0) \) to prove that there is overinvestment if the equilibrium is interior where \( \psi^{CE} (k_0) \) and \( \psi^{CP} (k_0) \) are as defined in Propositions 1 and 2. Consider equilibrium of Type 1 for both the banker from the Ramsey Problem and the Central Planner. If \( k \in [\hat{k}_0, k_0^{max}] \)

\[
\psi^{CP} (k_0) = z^{CP}_{1I} (k_0) \left[ 1 + \frac{\pi_l (\gamma - \theta - a_{1I})}{1 - \pi_h \theta [1 - \gamma]} - \frac{A + 1 - \theta \left[ \pi_l (1 - \theta) + \pi_h ((1 - \theta) (1 - \gamma) + a_{1I}) \right]}{1 - \pi_h \theta [1 - \gamma]} \right]
\]

where \( z^{CP}_{1I} = \frac{\bar{F}^{n} (k^C_{1I}) k^{C}_{1I} + A + 1 - \theta}{\bar{F}^{n} (k^C_{1I}) k^{C}_{1I} + \bar{F}^{n} (k^C_{1I}) - \theta} \). The expression for \( \psi^{CE} (k_0) \) is exactly the same as equation 62 with the only difference that \( z^{CP}_{1I} \) if replaced by \( z^{CE}_{1I} = \frac{A + 1 - \theta}{\bar{F}^{n} (k^C_{1I}) - \theta} \). From Lemma 1 and since \( z^{CP}_{1I} (k_0) - z^{CE}_{1I} (k_0) > 0 \)

\[
\psi^{CP} (k_0) - \psi^{CE} (k_0) = \left[ 1 + \frac{\pi_l (\gamma - \theta - a_{1I})}{1 - \pi_h \theta [1 - \gamma]} \right] (z^{CP}_{1I} (k_0) - z^{CE}_{1I} (k_0)) > 0
\]

It is clear that if the equilibrium is of Type 1 for the Central Planner and Type 2 for the banker, then there is overinvestment. Also if the equilibrium of the Central Planner is of type 2 there will be no overinvestment and one can easily show that the equilibrium will be of Type 2 for the banker as well.

**10.5.4 Decentralize the Constrained Central Planner**

**Proposition 4** Conditional on assumptions 1-6 and an interior equilibrium for the Central Planner (Type 1 equilibrium), the constrained Central Planner’s allocation can be decentralized using only a minimum capital requirement where the optimal minimum capital requirement is given by \( \rho^* = \frac{n}{k_0^{max}} \).

**Proof of Proposition 4.** The only relevant case is the interior equilibrium of Type 1. In the beginning of period zero, the policy maker chooses the optimal \( \rho \) assuming a symmetric equilibrium is played. In the equilibrium of Type 1, \( \lambda_{1h} > 0 \) and hence \( d_{1h} = \theta (1 - \gamma) k_0 \) and \( \lambda_{1l} = 0 \). Given that I proved that there is overinvestment in the Type 1 equilibrium, the minimum capital ratio constraint will always bind and hence if the policy maker chooses \( k_0 \), one can back out the optimal \( \rho^* \) from \( \rho^* = \frac{n}{k_0^{max}} \). In \( t = 0 \), the policy maker optimizes the ex-ante welfare of the consumers taking into account that \( k_{1h} \) is given by equation 44 and also \( d_{2s} = \theta k_{1s} \).

\[
\max_{k_0, k_{1l}} 3s + \pi_l \left[ \bar{F} (k^T_{1I}) - \bar{F}' (k^T_{1I}) k^T_{1I} - B_l - \delta (B_l) \right] + \pi_h (A + 1 - \theta) \frac{(1 - \theta) (1 - \gamma) + a_{1l} k_0}{(1 - \theta)} + \pi_l (A + 1 - \theta) k_{1l}
\]
subject to the period one budget constraint in the low state. From equation 45 and the budget constraint in period one in the low state

\[
k_{1t}\tilde{F}'(k_{1t}^T) + \frac{1}{\pi_t} [(1 - \pi_t\theta (1 - \gamma))k_{0} - n] \leq \left( \tilde{F}'(k_{1t}^T) + a_{1t} - \gamma \right)k_{0} + B_t + \theta k_{1t} \quad \left[ \pi_t z_{111}^{0,RP} \right]
\]

First order condition with respect to \(k_0\)

\[
\pi_t \left[ -\tilde{F}''(k_{1t}^T) k_{1t}^T \frac{\partial B_t}{\partial k_0} (1 + \delta'(B_t)) \right] + \pi_h (A + 1 - \theta) \frac{(1 - \theta) (1 - \gamma) + a_{1h}}{1 - \theta} \\
+ \pi_t z_{111}^{0,RP} \left[ \tilde{F}'(k_{1t}^T) + a_{1t} - \gamma + \tilde{F}''(k_{1t}^T) k_{1t}^T + \frac{\partial B_t}{\partial k_0} - \frac{(1 - \pi_h\theta (1 - \gamma))}{\pi_t} \right] = 0
\]

first order condition with respect to \(k_{1t}\)

\[
z_{111}^{0,RP} = \frac{A + 1 - \theta + \tilde{F}''(k_{1t}^T) k_{1t}^T - \frac{\partial B_t}{\partial k_{1t}} z_{111}^{1,RP}}{\tilde{F}''(k_{1t}^T) k_{1t}^T + \tilde{F}'(k_{1t}^T) - \theta - \frac{\partial B_t}{\partial k_{1t}}} \]

Plugging in for \(z_{111}^{1,RP}\)

\[
z_{111}^{1,RP} = z_{111}^{0,RP} = \frac{A + 1 - \theta + \tilde{F}''(k_{1t}^T) k_{1t}^T}{\tilde{F}''(k_{1t}^T) k_{1t}^T + \tilde{F}'(k_{1t}^T) - \theta} \quad (63)
\]

\[
-\pi_t F''(k_{1t}^T) k_{1t}^T + \pi_h (A + 1 - \theta) \frac{(1 - \theta) (1 - \gamma) + a_{1h}}{1 - \theta} \\
+ z_{111}^{0,RP} \left[ \pi_t \left( \tilde{F}'(k_{1t}^T) + \tilde{F}''(k_{1t}^T) k_{1t}^T - \theta + (\theta + a_{1t} - \gamma) \right) - (1 - \pi_h\theta (1 - \gamma)) \right] = 0 \quad (64)
\]

Equation 64 coincides with equation 54 (after plugging in for \(z_{0}^{CP} = z_{11}^{CP}\) and \(z_{1h}^{CP}\)) and equation 63 coincides with equation 55. As a result, \(k_{1t}^{T,CP} = k_{1t}^{T,CE}\) when the policy maker has an access to a minimum ex-ante capital requirement. Also both the banker and the Central Planner value wealth more in the crisis state relative to the high state in \(t = 1\). As a result, the optimal allocation of \(d_{ts}, k_0\) and \(k_{1s}\) coincide for the Ramsey Problem with ex-ante minimum capital requirement and the Central Planner’s problem. The optimal minimum capital requirement is \(\rho^* = \frac{\hat{n}}{\hat{c}_{0}^{CP}}\).

10.5.5 Smaller Fiscal Capacity Means Larger \(\rho\)

**Proposition 5** Conditional on assumptions 1-6 and an interior equilibrium for the Central Planner, smaller fiscal capacity implies optimally higher bank capital ratio \(\rho^*\), \(\frac{\partial \rho^*}{\partial \chi} > 0\)
Proof of Proposition 5. Consider an interior equilibrium for the Central Planner. By setting 
\[ \rho = \frac{n}{k_0} \] the policy maker can replicate the constrained Central Planner’s allocation. Since I proved that \( \frac{\partial \psi_{\text{CP}} (k_0; x)}{\partial x_0} > 0 \), it is sufficient to prove that holding \( k_0 \) constant, \( \frac{\partial \psi_{\text{CP}} (x; k_0)}{\partial x} > 0 \) (partial derivative) in order to prove that \( \frac{\partial k_0^{\text{CP}}}{\partial x} < 0 \). Define

\[
H (x; k_0) = (\theta + a_{1l} - \gamma) k_0 + k_{1l}^T \left( \tilde{F}' (k_{1l}^T) - \theta \right) + B_l (k_{1l}^T, \chi) - \frac{1}{\pi_l} \left[ k_0 \left[ 1 - \pi_h \theta (1 - \gamma) \right] - n \right] = 0
\]

For a given \( k_0 \) (partial derivative), totally differentiate \( H (x; k_0) = 0 \) with respect to \( x \)

\[
\frac{\partial k_{1l}^T (x; k_0)}{\partial x} = -\frac{\partial B_l (x; k_{1l}^T)}{\partial x} \left( \tilde{F}' (k_{1l}^T) - \theta + \tilde{F}'' (k_{1l}^T) k_{1l}^T + \frac{\partial B_l (k_{1l}^T; x)}{\partial x_{k_{1l}^T}} \right) > 0
\]

where \( \frac{\partial B_l (x; k_{1l}^T)}{\partial x} < 0 \) is the partial derivative of \( B_l \) with respect to \( x \), holding \( k_{1l}^T \) constant. I can solve for \( \frac{\partial B_l (x; k_{1l}^T)}{\partial x} \) by totally differentiating the first order condition that pins down \( B_l \), holding \( k_{1l}^T \) constant. The derivative is given by \( 0 = \frac{\partial \delta (B_l)}{\partial B_l} \frac{\partial B_l (x; k_{1l}^T)}{\partial x} + \frac{\partial \delta (B_l)}{\partial B_l} \frac{\partial B_l (x; k_{1l}^T)}{\partial x} \). Since \( \frac{\partial \delta (B_l)}{\partial B_l} > 0 \) and \( \frac{\partial \delta (B_l)}{\partial B_l} > 0 \),

\[
\frac{\partial B_l (x; k_{1l}^T)}{\partial x} = \frac{\partial \delta (B_l)}{\partial B_l} \frac{\partial \delta (B_l)}{\partial B_l} = -\frac{B_l}{(\eta - 1)x} < 0.
\]

Also for a given \( \chi \) as shown in Lemma 1 \( \frac{\partial B_l (k_{1l}^T; x)}{\partial k_{1l}^T} = 0 \). The fact that \( \frac{\partial k_{1l}^T (x; k_{0})}{\partial x} > 0 \) is intuitive and means that for a given level period zero investment, the fire sale will be larger for the country with the smaller fiscal capacity.

\[
\frac{\partial \psi_{\text{CP}} (x; k_0)}{\partial x} = \frac{\partial z_{1l}^{\text{CP}}}{\partial x} - \frac{\partial z_0^{\text{CP}}}{\partial x} = \left[ 1 + \frac{\pi_l (\gamma - \theta - a_{1l})}{1 - \pi_h \theta (1 - \gamma)} \right] \frac{\partial k_{1l}^T (x; k_0)}{\partial x} \frac{\partial z_0^{\text{CP}}}{\partial k_{1l}^T} > 0
\]

where I proved that \( \frac{\partial z_0^{\text{CP}}}{\partial k_{1l}^T} > 0 \) in Lemma 1 since \( z_{1l}^{\text{RP}} = z_{1l}^{\text{CP}} \) if the equilibrium is of Type 1. This completes the proof.

10.5.6 Proposition 6

Proposition 6 Conditional on Assumptions 1-6, considering a symmetric equilibrium for the Ramsey Problem with \( N \) banks, no commitment, no minimum capital requirement and no minimum liquidity requirement, there is never a fire sale in the high state, \( q_{1h} = 1 \) and there is a fire sale in the low state, \( q_{1l} < 1 \). Conditional on the assumptions that \( F'' (k_{1l}^T) = 0 \) and

\[
(1 + \frac{1}{N}) F'' (k_{1l}^T) - \left[ \frac{\delta'' (B_l) \left( 1 - 2 z_{1l}^{\text{RP}} \right)}{\delta'' (B_l)} \right]^2 + \frac{1}{(F'' (k_{1l}^T) k_{1l}^T + F' (k_{1l}^T - \theta)^2 \delta'' (B_l)) N} < 0
\]

Assumption 8

where \( z_{1l}^{\text{RP}} = \frac{F'' (k_{1l}^T) k_{1l}^T + A + 1 - \theta}{F'' (k_{1l}^T) k_{1l}^T + F' (k_{1l}^T - \theta)^2} \) and \( B_l = (\delta')^{-1} \left( z_{1l}^{\text{RP}} - 1 \right) \), the equilibrium is unique and exists. Also \( z_{1l}^{\text{CE,N}} = z_{0\text{CE,N}} \).
Proof of Proposition 6. The first part of the proof regarding \( q_{1h} = 1 \) and \( q_{1l} < 1 \) is exactly the same as in Proposition 1 given that when there is no fire sale in the low state, \( q_{1l} = 1 \), \( \bar{F}'' (k_{1l}^T) = 0 \) and \( \frac{1}{\delta''(B_i)N} \frac{\partial z_{1l,RP}^{1,RP}}{\partial k_{1l}^T} = 0 \). Therefore I omit it. Next I prove that the equilibrium is unique and exists and if \( \rho = 0 \), it can be one of the two Types: Type 1: \( z_{1l}^{CE,N} = z_{0}^{CE,N} > z_{1h}^{CE,N} \) and Type 2: \( z_{0}^{CE,N} > z_{1s}^{CE,N} \).

First let’s consider the case \( \lambda_{1s}^{CE,N} = 0 \left( z_{1l}^{CE,N} = z_{0}^{CE,N} = z_{1l}^{CE,N} \right) \) and prove that given Assumption 2 this case will never be an equilibrium. If \( \varphi = 0 \) (no minimum liquidity requirement) and \( \rho = 0 \) (no minimum bank capital requirement) one can re-write the First order conditions with respect to \( k_{1s} \) and \( k_{0} \)

\[
\frac{A + 1 - \theta}{1 - \bar{F}''(k_{1s}^T) k_{1s}^T} = \frac{A + 1 - \theta}{(1 - \theta)} = z_{1h}^{CE,N}
\]

(65)

\[
1 - \bar{F}'(k_{1l}^T) = \frac{1}{N} \left[ \bar{F}''(k_{1l}^T) k_{1l}^T + \frac{1}{\delta''(B_i)N} \frac{\partial z_{1l,RP}^{1,RP}}{\partial k_{1l}^T} \right]
\]

\[
z_{0}^{CE,N} = \pi_l (A + 1 - \theta) + \pi_l z_{1l}^{CE,N} (\theta + a_{1l} - \gamma) + \pi_h \frac{A + 1 - \theta}{(1 - \theta)} (1 + a_{1h} - \gamma)
\]

(66)

Plugging in \( z_{1l}^{CE,N} = z_{1l}^{CE,N} = z_{0}^{CE,N} = \frac{A+1+\theta}{(1-\theta)} \) in equation 66 and simplifying implies \( 1 = \sum \pi_s (1 + a_{1s} - \gamma) \). However, given Assumption 2, the equation above will not be satisfied and hence \( z_{1h}^{CE,N} = z_{0}^{CE,N} = z_{1l}^{CE,N} \) will not be an equilibrium given the assumptions made.

Type 1 equilibrium: \( \lambda_{1h}^{CE,N} > 0, \lambda_{1l}^{CE,N} = 0 \left( z_{1l}^{CE,N} = z_{0}^{CE,N} > z_{1h}^{CE,N} \right) \)

Using the fact that \( \lambda_{1h}^{CE,N} = z_{0}^{CE,N} - z_{1h}^{CE,N} \), one can show that \( z_{1h}^{CE,N} = \frac{A+1+\theta}{(1-\theta)} \) and

\[
z_{0}^{CE,N} = \frac{\pi_h \frac{A+1-\theta}{(1-\theta)} ((1-\theta)(1-\gamma) + a_{1h}) + \pi_l (A+1-\theta) + z_{1l}^{CE,N} \pi_l (\theta + a_{1l} - \gamma)}{1 - \pi_h \theta (1 - \gamma)}
\]

(67)

\[
z_{1l}^{CE,N} = \frac{A + 1 - \theta}{\frac{1}{N} \bar{F}''(k_{1l}^T) k_{1l}^T + \frac{1}{\delta''(B_i)N} \frac{\partial z_{1l,RP}^{1,RP}}{\partial k_{1l}^T} + \bar{F}'(k_{1l}^T) - \theta}
\]

(68)

One can solve for \( k_{1l}^T \) and from there for all other endogenous variables by plugging in \( z_{1l}^{CE,N} \) and \( z_{1l}^{CE,N} \) in the first order condition with respect to \( k_{0} \), equation 30, and taking into account that \( \lambda_{1h}^{CE,N} > 0 \) and \( \lambda_{1l}^{CE,N} = 0 \). In order for \( z_{1l}^{CE,N} > z_{1l}^{CE,N} \), it has to be the case that \( z_{1l}^{CE,N} > \frac{A+1+\theta}{(1-\theta)} \) which implies

\[
\frac{1}{N} \bar{F}''(k_{1l}^T) k_{1l}^T + \frac{1}{\delta''(B_i)N} \frac{\partial z_{1l,RP}^{1,RP}}{\partial k_{1l}^T} + \bar{F}'(k_{1l}^T) - 1 < 0
\]

The rest of the equations are the same as in the equilibrium of Type 1 in the CE with a continuum of banks (see the proof of Proposition 1).

Type 2 equilibrium: \( \lambda_{1s}^{CE,N} > 0 \left( z_{0}^{CE,N} > z_{1s}^{CE,N} \right) \)
It will be the case that \( \lambda^C_{1s} = z^C_0 - z^C_{1s} \) and using the First order conditions with respect to \( k_0 \) and \( k_{1s} \), equations 30 and 31, one can solve for \( z^C_{1s} \) and \( z^C_0 \) as a function of \( k^T_N \). The rest of the equations are the same as in the equilibrium of Type 2 in the CE with a continuum of banks (see the proof of Proposition 1).

Finally, I show that the case \( \lambda^C_{11} = 0, \lambda^C_{1l} > 0 \left( z^C_1 = z^C_0 > z^C_{1l} \right) \) is impossible. \( \lambda^C_{1l} = z^C_0 - z^C_{1l} \) and \( z^C_{1l} = \frac{A + 1 - \theta}{1 - \theta} \) and

\[
\begin{align*}
\lambda^C_{11} &= \frac{A + 1 - \theta}{1 - \theta} - z^C_{1l} \\
\lambda^C_{1l} &= z^C_0 - z^C_{1l} \\
z^C_{1l} &= \left( \frac{A + 1 - \theta}{1 - \theta} \right) k_0 - \frac{1}{N} \tilde{F}'' \left( k^T_N \right) k_{1l} + \frac{1}{N} \frac{\partial F' \left( k^T_N \right)}{\partial k_{1l}} + \tilde{F}' \left( k^T_N \right) - \theta - \frac{1}{N} \tilde{F}'' \left( k^T_N \right) k_0 
\end{align*}
\]

In order for \( z^C_{1l} > z^C_{1l} \), it will have to be the case \( A + 1 - \theta \frac{1}{1 - \theta} \) \( z^C_{1l} > \frac{A + 1 - \theta}{1 - \theta} \frac{1}{N} \tilde{F}'' \left( k^T_N \right) k_{1l} + \frac{1}{N} \frac{\partial F' \left( k^T_N \right)}{\partial k_{1l}} + \tilde{F}' \left( k^T_N \right) - \theta \)

which implies

\[
\frac{1}{N} \tilde{F}'' \left( k^T_N \right) k_{1l} + \frac{1}{N} \frac{\partial F' \left( k^T_N \right)}{\partial k_{1l}} + \tilde{F}' \left( k^T_N \right) - 1 > 0
\]

Rewriting the first order condition with respect to \( k_0 \)

\[
z^C_{00} = \left[ \frac{\pi_h^C z^C_{1h} \left( 1 + a_{1h} - \gamma \right) \varpi \left( k^T_N \right) + \pi_l \left( A + 1 - \theta - z^C_{1h} \right) \frac{1}{N} \tilde{F}'' \left( k^T_N \right) k_0 \left( \varpi \left( k^T_N \right) + \theta + a_{1l} - \gamma - \theta \left( \tilde{F}' \left( k^T_N \right) - \gamma \right) \right) \right]}{\varpi \left( k^T_N \right) \left[ 1 - \pi_l \theta \left( \tilde{F}' \left( k^T_N \right) - \gamma \right) + \frac{1}{N} \tilde{F}'' \left( k^T_N \right) k_0 \right]}
\]

where \( \varpi \left( k^T_N \right) = \frac{1}{N} \tilde{F}'' \left( k^T_N \right) k_{1l} + \frac{1}{N} \frac{\partial F' \left( k^T_N \right)}{\partial k_{1l}} + \tilde{F}' \left( k^T_N \right) - \theta - \frac{1}{N} \tilde{F}'' \left( k^T_N \right) k_0 \).

Next I prove that given Assumption 2 and the inequality 70 it will be always the case that \( z^C_{00} > z^C_{1h} \) which will imply that the case \( z^C_{1h} = z^C_{00} > z^C_{11} \) is impossible. Re-writing \( z^C_{00} > z^C_{1h} \) \( \frac{A + 1 - \theta}{1 - \theta} \)

\[
\begin{align*}
&\left\{ \frac{1}{N} \tilde{F}'' \left( k^T_N \right) k_{1l} + \frac{1}{N} \frac{\partial F' \left( k^T_N \right)}{\partial k_{1l}} + \tilde{F}' \left( k^T_N \right) - 1 \right\} \left\{ \sum_s \pi_s \left( 1 + a_{1s} - \gamma \right) \right. \\
&\quad + \pi_l \left( \gamma - a_{1l} - \theta \right) + \pi_l \theta \left( \tilde{F}' \left( k^T_N \right) - \gamma \right) \right. \\
&\quad + \left. \left\{ \sum_s \pi_s \left( 1 + a_{1s} - \gamma \right) - 1 \right\} - 1 - \theta - \frac{1}{N} \tilde{F}'' \left( k^T_N \right) k_0 \right\} > 0
\end{align*}
\]

This completes the proof that \( z^C_{00} > z^C_{1h} \). As a result, the case \( \lambda^C_{1h} = 0, \lambda^C_{1l} > 0 \) will never be an equilibrium. Therefore, it will be sufficient to only consider the interior equilibrium of Type 1 when proving existence and uniqueness.

Existence and Uniqueness

65
As in the proof of Proposition 1, one can prove that for every $k_0 \in [0, k_{0}^{\text{max}}]$, there exists an unique $k_{l1}^{T}$ and if the equilibrium $k_0$ is such that $k_0 \in (\hat{k}_0, k_{0}^{\text{max}}]$ then there will be a fire sale $k_{l1}^{T} > 0$ and $k_{l1}^{T}$ is unique where $k_0$ is pinned down by equation 47. Also one can prove that $\frac{\partial k_{l1}^{T}(k_0)}{\partial k_0} > 0$. Following the same steps as in the Proof of Proposition 1, define the following function which will be used to pin down the equilibrium $b_0$

$$\psi^{CE,N}(k_{l1}^{T}(k_0)) = z_{l1}^{CE,N}(k_{l1}^{T}(k_0)) - z_{0}^{CE,N}(k_{l1}^{T}(k_0))$$ if $k_{l1}^{T} \in [\hat{k}_{l1}, \hat{k}_{l1}^{T,\text{max}}]$

where $\hat{k}_{l1}^{T,\text{max}}$ is pinned down by equation 33. Also as before, given Assumption 6, $\hat{k}_{l1}^{T,\text{max}}$ exists and is unique (the proof is the same as the proof in Proposition 1). $\hat{k}_{l1}^{T}$ is pinned down by $M(\hat{k}_{l1}^{T}) = 0$ if $0 < \hat{k}_{l1}^{T} < \hat{k}_{l1}^{T,\text{max}}$ where

$$M(k_{l1}^{T}) = \frac{1}{N} \bar{F}''(k_{l1}^{T}) k_{l1}^{T} - \frac{\partial B_l}{\partial k_0} + \bar{F}'(k_{l1}^{T}) - 1$$

Given Assumption 8, one can prove that $M'(\hat{k}_{l1}^{T}) < 0$. If $M(\cdot) \leq 0$ for every $k_{l1}^{T} \in [0, \hat{k}_{l1}^{T,\text{max}}]$, then $\hat{k}_{l1}^{T} = 0$. If $M(\cdot) \geq 0$ for every $k_{l1}^{T} \in [0, \hat{k}_{l1}^{T,\text{max}}]$ then $\hat{k}_{l1}^{T} = \hat{k}_{l1}^{T,\text{max}}$. Otherwise $\hat{k}_{l1}^{T}$ is pinned down by $M(\hat{k}_{l1}^{T}) = 0$ and $\hat{k}_{l1}^{T}$ is unique since $M'(\hat{k}_{l1}^{T})$ is strictly decreasing. Notice that if there is no moral hazard, $N \to \infty$ or $\chi \to \infty$, then $M(0) = 0$ and $\hat{k}_{l1}^{T} = 0$. Since $M$ is strictly decreasing, if $M < 0$, the equilibrium is of Type 1. Given that I proved that only the equilibrium of Type 1 is possible, the relevant range for $k_{l1}^{T}$ to consider is $[\hat{k}_{l1}^{T}, \hat{k}_{l1}^{T,\text{max}}]$. Consider the interior equilibrium of Type 1. Given Assumption 8 which implies $\frac{\partial}{\partial k_{l1}^{T}(k_0)} \left[ \frac{1}{N} \bar{F}''(k_{l1}^{T}) k_{l1}^{T} + \frac{\partial B_l}{\partial k_0} + \bar{F}'(k_{l1}^{T}) - \theta \right] > 0$, from equation 68

$$\frac{\partial z_{l1}^{CE,N}(k_0)}{\partial k_0} = -\frac{\partial}{\partial k_{l1}^{T}(k_0)} \left[ \frac{1}{N} \bar{F}''(k_{l1}^{T}) k_{l1}^{T} + \frac{\partial B_l}{\partial k_0} + \bar{F}'(k_{l1}^{T}) \right] + \frac{\partial k_{l1}^{T}(k_0)}{\partial k_0} \frac{\partial z_{l1}^{CE,N}(k_0)}{\partial k_{l1}^{T}(k_0)} > 0$$

Combining equations 68 and 67

$$\frac{\partial \psi^{CE,N}(k_{l1}^{T}(k_0))}{\partial k_0} = \frac{\partial z_{l1}^{CE,N}(k_0)}{\partial k_0} - \frac{\partial z_{0}^{CE,N}(k_0)}{\partial k_0} = \frac{\partial z_{l1}^{CE,N}(k_0)}{\partial k_0} \left[ 1 + \frac{\pi (\gamma - \theta - a_{1l})}{1 - \pi \theta (1 - \gamma)} \right] > 0$$

Given that $\psi^{CE,N}(k_0)$ is strictly increasing, if the equilibrium exists, it is unique. In order to prove existence, evaluate the $\psi^{CE,N}(\hat{k}_{l1}^{T})$ function at $k_{l1}^{T} = \hat{k}_{l1}^{T}$ and $k_{l1}^{T} = \hat{k}_{l1}^{T,\text{max}}$ which is the relevant range for the Type 1 equilibrium. If $k_{l1}^{T} = \hat{k}_{l1}^{T}$, then $z_{l1}^{CE,N}(\hat{k}_{l1}^{T}) = \frac{A + 1 - \theta}{1 - \theta} \frac{1}{1 - \pi \theta (1 - \gamma)} < 0$
If $\psi^{CE,N}(k_{1i}^{T,\text{max}}) < 0$ for every $k_{1i}^T \in [k_{1i}^T, k_{1i}^{T,\text{max}}]$, then the equilibrium is a corner equilibrium of Type 2. If $\psi^{CE,N}(k_{1i}^{T,\text{max}}) > 0$, the equilibrium is an interior equilibrium of Type 1. This completes the proof of existence.

10.5.7 Proposition 7 and Corollary 1

Corollary 1 If $1 < N < \infty$ and $\chi < \infty$, there is moral hazard, i.e. $\frac{\partial B_i^l}{\partial k_0^l} = \frac{1}{\delta''(B_i^l)N} \frac{\partial z_{1i}^{1,RP}(k_{1i}^T) \partial k_{1i}^{i,T}}{\partial d_{1i}^l} > 0$ and $\frac{\partial B_i^l}{\partial d_{1i}^l} = \frac{1}{\delta''(B_i^l)N} \frac{\partial z_{1i}^{1,RP}(k_{1i}^T) \partial k_{1i}^{i,T}}{\partial d_{1i}^l} > 0$. Also for a given $k_{1i}^T$, the fewer the banks are and the larger the fiscal capacity is, the larger the moral hazard is $\frac{\partial B_i^l}{\partial k_0^l N} < 0$, $\frac{\partial B_i^l}{\partial k_0^l \chi} < 0$ and $\frac{\partial B_i^l}{\partial d_{1i}^l N} < 0$, $\frac{\partial B_i^l}{\partial d_{1i}^l \chi} < 0$.

Proof of Corollary 1. Totally differentiate the budget constraint in the low state given by equation 29 with respect to $k_0^l$ holding $d_{1i}^l$ fixed to solve for $\frac{\partial k_{1i}^l}{\partial k_0^l}$

$$\frac{\partial k_{1i}^l}{\partial k_0^l} = \frac{\frac{\partial B_i^l}{\partial k_0^l} - 1}{\left(\tilde{F}'(k_{1i}^T) + \frac{1}{N} \tilde{F}''(k_{1i}^T) k_{1i}^{i,T} - \theta\right)}$$

(72)

Totally differentiate the budget constraint in the low state with respect to $d_{1i}^l$ holding $k_0^l$ fixed to solve for $\frac{\partial k_{1i}^l}{\partial d_{1i}^l}$

$$\frac{\partial k_{1i}^l}{\partial d_{1i}^l} = \frac{\frac{\partial B_i^l}{\partial d_{1i}^l}}{\left(\tilde{F}'(k_{1i}^T) + \frac{1}{N} \tilde{F}''(k_{1i}^T) k_{1i}^{i,T} - \theta\right)}$$

(73)

From the first order condition with respect to $B_i$ of the policy maker in the middle period

$$\frac{\partial B_i^l}{\partial k_0^l} = \frac{1}{\delta''(B_i^l)N} \frac{\partial z_{1i}^{1,RP}(k_{1i}^T) \partial k_{1i}^{i,T}}{\partial d_{1i}^l} = \frac{1}{\delta''(B_i^l)N} \frac{\partial z_{1i}^{1,RP}(k_{1i}^T)}{\partial k_{1i}^{i,T}} \left(1 - \frac{\partial k_{1i}^l}{\partial k_0^l}\right)$$

(74)

$$\frac{\partial B_i^l}{\partial d_{1i}^l} = \frac{1}{\delta''(B_i^l)N} \frac{\partial z_{1i}^{1,RP}(k_{1i}^T) \partial k_{1i}^{i,T}}{\partial d_{1i}^l} = -\frac{\partial k_{1i}^l}{\partial d_{1i}^l} \frac{1}{\delta''(B_i^l)N} \frac{\partial z_{1i}^{1,RP}(k_{1i}^T)}{\partial k_{1i}^{i,T}}$$

(75)

Combining equations 72 and 74, and also equations 73 and 75, and also from Assumption 4

$$\frac{\partial B_i^l}{\partial k_0^l} = \frac{\partial z_{1i}^{1,RP}(k_{1i}^T)}{\partial k_{1i}^{i,T}} \left(\gamma - \frac{\partial z_{1i}^{1,RP}(k_{1i}^T)}{\partial k_{1i}^{i,T}} \left(\tilde{F}'(k_{1i}^T) - \theta + \frac{1}{N} \tilde{F}''(k_{1i}^T) k_{1i}^{i,T} - \theta\right)\right) > 0$$

$$\frac{\partial B_i^l}{\partial d_{1i}^l} = \frac{\partial z_{1i}^{1,RP}(k_{1i}^T)}{\partial k_{1i}^{i,T}} \left(\tilde{F}'(k_{1i}^T) - \theta + \frac{1}{N} \tilde{F}''(k_{1i}^T) k_{1i}^{i,T} - \theta\right) > 0$$
Also notice that for a given $k^T_{1l}$, the fewer the banks are and the larger the fiscal capacity is, the larger the moral hazard is $\frac{\partial B^i_{1l}}{\partial k^T_{1l} \partial N} < 0$, $\frac{\partial B^i_{1l}}{\partial k^T_{1l} \partial \chi} < 0$ and $\frac{\partial B^i_{1l}}{\partial d^T_{1l} \partial N} < 0$, $\frac{\partial B^i_{1l}}{\partial d^T_{1l} \partial \chi} < 0$. The comparative statics with respect to fiscal capacity follows from the fact that the larger $\chi$ is, the larger $\delta'' (B^i_l)$ is.

**Proposition 7** Conditional on Assumptions 1-6 and 8 and also assuming $A > 1 - \theta$ and $N > 1$, comparing the constrained Central Planner’s allocation without commitment and the Competitive Equilibrium from the Ramsey Problem with $N$ banks with no commitment, no minimum capital requirement and no minimum liquidity requirement, there is always overinvestment, $k^CP_0 < k^{CE,N}_0$, if the equilibrium is of Type 1 for the Central Planner (interior equilibrium) and there is no overinvestment, $k^CP_0 = k^{CE,N}_0$, if the equilibrium is of Type 2 for the Central Planner (corner equilibrium).

**Proof of Proposition 7.** Consider the equilibrium of Type 1 for both the Central Planner and the banker from the Ramsey Problem. Since $\psi^{CE,N}(k_0) > 0$ and $\psi^{CP}(k_0) > 0$, it is sufficient to prove that $\psi^{CP}(k_0) > \psi^{CE,N}(k_0)$ in order to prove that there is overinvestment if the equilibrium is interior. $\psi^{CP}(k_0)$ is given by equation 62 where $z^{CP}_l$ is given by equation 55. The expression for $\psi^{CE,N}(k_0)$ is the same as equation 62 with the only difference that $z^{CP}_l$ is replaced by $z^{CE,N}_l$ which is given by equation 68. First I prove that given the assumptions made, $z^{CP}_l - z^{CE,N}_l > 0$

$$z^{CP}_l (k_0) - z^{CE,N}_l (k_0) = \tilde{F}'' (k^T_{1l}) k^T_{1l} \left( \frac{1}{N} \tilde{F}'' (k^T_{1l}) k^T_{1l} + \tilde{F}' (k^T_{1l}) - \theta \right)$$

$$+ (A + 1 - \theta) \left( \frac{1}{N} - 1 \right) \tilde{F}'' (k^T_{1l}) k^T_{1l} + \left( A + 1 - \theta + \tilde{F}'' (k^T_{1l}) k^T_{1l} \right) \frac{1}{\delta'' (B_l) N} \frac{\partial z^{1,RP}_{1l}}{\partial k^T_{1l}} > 0$$

Given Assumption 4 $\left( A + 1 - \theta + \tilde{F}'' (k^T_{1l}) k^T_{1l} \right) > 0$ and also from Lemma 2 $\frac{1}{\delta'' (B_l) N} \frac{\partial z^{1,RP}_{1l}}{\partial k^T_{1l}} \geq 0$, it is sufficient to prove that

$$(A + 1 - \theta) \left( 1 - \frac{1}{N} \right) > 1 - \theta > \left( \tilde{F}' (k^T_{1l}) - \theta + \tilde{F}'' (k^T_{1l}) k^T_{1l} \right) \frac{k^T_{1l}}{N}$$

which implies $A (N - 1) > (1 - \theta)$. If $N \geq 2$ and $A > 1 - \theta$, equation 76 is satisfied

$$\psi^{CP}(k_0) - \psi^{CE,N}(k_0) = \left[ 1 + \frac{\pi_l (\gamma - \theta - a_M)}{1 - \pi_l \theta (1 - \gamma)} \right] \left( z^{CP}_l (k_0) - z^{CE,N}_l (k_0) \right) > 0$$

It is clear that if the equilibrium is of Type 1 for the Central Planner and Type 2 for the banker then there is overinvestment. Also if the equilibrium of the Central Planner is of type 2 there will be no overinvestment and one can easily show that the equilibrium will be of Type 2 for the banker as well.
10.5.8 Proposition 8

Proposition 8 Given Assumptions 1-6,8, for a given exogenous $\hat{\rho} > \frac{n}{k_0^{CE,N}}$, and assuming no ex ante minimum liquidity requirement, considering a symmetric equilibrium, the CE from the Ramsey Problem with $N$ banks can be one of the following four types:

**Proposition 11** 1) Type 1: $z_{11}^{CE,N} = z_{0}^{CE,N} > z_{1h}^{CE,N}$ if $k_{11}^{T} \in [\hat{k}_{11}^{T}, \hat{k}_{11}^{T,max})$

2) Type 2: $z_{0}^{CE,N} > z_{1h}^{CE,N}$ if $k_{11}^{T} = \hat{k}_{11}^{T,max}$

3) Type 3: $z_{11}^{CE,N} = z_{0}^{CE,N} = z_{1h}^{CE,N}$ if $k_{11}^{T} = \hat{k}_{11}^{T}$

4) Type 4: $z_{11}^{CE,N} = z_{0}^{CE,N} > z_{1h}^{CE,N}$ if $k_{11}^{T} \in [0, \hat{k}_{11}^{T}]$

$\hat{k}_{11}^{T,max}$ is pinned down by equation 33 in the Appendix. $\hat{k}_{11}^{T}$ is unique and exists and if $0 < \hat{k}_{11}^{T} < \hat{k}_{11}^{T,max}$, then $\hat{k}_{11}^{T}$ is pinned down by $M(\hat{k}_{11}^{T}) = 0$ where

$$M(k_{11}^{T}) = \frac{1}{N} \hat{F}'(k_{11}^{T}) k_{11}^{T} + \frac{1}{\delta''(B_{l}) N} \frac{\partial z_{11}^{1,RP}}{\partial k_{11}^{T}} + \hat{F}'(k_{11}^{T}) - 1.$$  

$k_{0}^{CE,N}$ is the optimal period zero investment chosen by the banker if there is no minimum capital requirement or liquidity requirement.

**Proof of Proposition 8.** The proof that $\hat{k}_{11}^{T}$ is unique and exists is provided in the proof of Proposition 6. From the minimum capital requirement constraint binding $k_{0}$ is pinned down by $k_{0} = \frac{n}{\hat{\rho}}$.

**Equilibrium of Type 1:** If $\lambda_{1h}^{CE,N} > 0, \lambda_{11}^{CE,N} = 0 \left(z_{11}^{CE,N} = z_{0}^{CE,N} > z_{1h}^{CE,N}\right)$

Notice that $\lambda_{1h}^{CE,N} = z_{0}^{CE,N} - z_{1h}^{CE,N} > 0$. Since the minimum capital ratio constraint is always binding, $z_{0}^{CE,N}$ is pinned down by $z_{0}^{CE,N} = z_{11}^{CE,N}$. From the budget constraints in $t = 1$ and using the fact that the period 2 collateral constraints and the period one collateral constraint in the high state are binding, the following equation determines $k_{11}^{T}$

$$\left(\frac{n}{\hat{\rho}} - k_{11}^{T}\right) \left(\hat{F}'(k_{11}^{T}) - \theta\right) + \frac{1}{\pi_{l}} \left[(1 - \pi_{h} \theta (1 - \gamma)) \frac{n}{\hat{\rho}} - n\right] = \left(\hat{F}'(k_{11}^{T}) + a_{l} - \gamma\right) \frac{n}{\hat{\rho}} + B_{l}(k_{11}^{T})$$

The rest of the endogenous variables $b_{0}, k_{1s}$ and $d_{s}$ are pinned down by equations 43,44,45 and 46.

Re-writting the First order conditions with respect to $k_{1s}$, $z_{1s}^{CE,N} = \frac{A+1-\theta}{1-\theta}$ where $z_{1s}^{CE,N}$ is pinned down by equation 68. In order for $z_{1s}^{CE,N} > z_{1h}^{CE,N}$

$$M(k_{11}^{T}) = \frac{1}{N} \hat{F}'(k_{11}^{T}) k_{11}^{T} + \frac{1}{\delta''(B_{l}) N} \frac{\partial z_{11}^{1,RP}}{\partial k_{11}^{T}} + \hat{F}'(k_{11}^{T}) - 1 < 0 \quad (77)$$

As long as the optimal $k_{11}^{T}$ is in the set defined by inequality 77, the equilibrium is of Type 1.

**Equilibrium of Type 2:** If $\lambda_{1s}^{CE,N} > 0 \left(z_{0}^{CE,N} > z_{1s}^{CE,N}\right)$
This will be the optimal equilibrium only if \( \hat{\rho} = \frac{n}{n_{\text{max}}} \). The rest of the equations are the same as the equations in the equilibrium of Type 2 in Proposition 1.

**Equilibrium of Type 3:** If \( \lambda_{ls}^{CE,N} = 0 \left( z_{1l}^{CE,N} = z_0^{CE,N} = z_{1h}^{CE,N} \right) \)

\[ z_{1h}^{CE,N} = \frac{A+1-\theta}{(1-\theta)} \] and \( z_{1l}^{CE,N} \) is pinned down by equation 68. The optimal fire sale in this type of equilibrium is \( \hat{k}_{1l}^T \) and it is determined by \( z_{1l}^{CE,N} = z_{1l}^{CE,N} \) which simplifies to \( M \left( \hat{k}_{1l}^T \right) = 0 \) where \( M \left( \hat{k}_{1l}^T \right) \) is given by equation 77. Given that \( k_0 = \frac{n}{\hat{\rho}} \), \( k_{1l} = k_0 - \hat{k}_{1l}^T \). From the budget constraint in the low state

\[ d_{1l} = \left( \tilde{F}' \left( \hat{k}_{1l}^T \right) + a_{1l} - \gamma \right) k_0 + B_l + \left( \theta - \tilde{F}' \left( \hat{k}_{1l}^T \right) \right) \left( k_0 - \hat{k}_{1l}^T \right) \]

Using the fact that \( k_0 - n = \sum_s \pi_s d_{1s} \)

\[ d_{1h} = \frac{k_0 - n - \pi_l d_{1l}}{\pi_h} \]

and using the budget constraint in the high state \( k_{1h} = \frac{(1+a_{1h}-\gamma)k_0 - d_{1h}}{d_{2s}} \) where \( d_{2s} = \theta k_{1s} \).

**Equilibrium of Type 4:** If \( \lambda_{1h}^{CE,N} = 0, \lambda_{1l}^{CE,N} > 0 \left( z_{1h}^{CE,N} = z_0^{CE,N} > z_{1l}^{CE,N} \right) \)

From the budget constraint in the low state, I can solve for \( k_{1l}^T \) using the fact \( k_0 = \frac{n}{\hat{\rho}} \)

\[ (k_0 - k_{1l}^T) \left( \tilde{F}' \left( k_{1l}^T \right) - \theta \right) = \left( a_{1l} + (1 - \theta) \left( \tilde{F}' \left( k_{1l}^T \right) - \gamma \right) \right) k_0 + B_l \]

Also \( k_{1l} = k_{1l}^T - k_0 \), \( d_{2s} = \theta k_{1s} \), \( d_{1l} = \theta \left( \tilde{F}' \left( k_{1l}^T \right) - \gamma \right) k_0 \) and

\[ d_{1h} = \left[ \left( 1 - \pi_l \theta \left( \tilde{F}' \left( k_{1l}^T \right) - \gamma \right) \right) k_0 - n \right] \frac{1}{\pi_h} \]

\[ k_{1h} = \frac{(1+a_{1h}-\gamma)k_0 - d_{1h}}{d_{2s}} \]

\[ z_{1h}^{CE,N} = \frac{A+1-\theta}{(1-\theta)} \] and \( z_{1l}^{CE,N} \) is given by equation 69. \( z_{1h}^{CE,N} > z_{1l}^{CE,N} \) implies that \( k_{1l}^T \) determined by \( k_{1l}^T = z_0^{CE,N} \) has to be such that it is in the set defined by \( M \left( k_{1l}^T \right) > 0 \) in order for the equilibrium to be of Type 4.

10.5.9 Proposition 9

**Proposition 9** Conditional on the policy maker having access to two ex-ante instruments — an ex-ante tax on period zero investment ("price" instrument), \( \tau_{k_0} \), and a minimum liquidity requirement, one can show that \( \tau_{k_0} > 0 \). If \( N < \infty \) (no moral hazard) then \( \frac{\partial \tau_{k_0}}{\partial x} = 0 \). If \( 1 < N < \infty \text{ and } A > 1-\theta \) then \( \frac{\partial \tau_{k_0}}{\partial x} < 0 \). \( \tau_{k_0} \) and \( \frac{\partial \tau_{k_0}}{\partial x} \) are given by

\[ \tau_{k_0} = \left[ \frac{z_{1l}^{CP} \left( k_{1l}^T \right)}{z_{1l}^{CE,N} \left( k_{1l}^T, \chi \right)} - 1 \right] \Phi > 0 \]
where $\Phi = \frac{[1 - \pi_h \theta (1 - \gamma) + \pi_t (\gamma - \theta - a_{11})]}{(1 - \pi)} > 0$.

**Proof of Proposition 9.** Conditional on the policy maker having access to two ex-ante instruments — an ex-ante tax on period zero investment ("price" instrument), $\tau$, and a minimum liquidity requirement, the constrained Central Planner's allocation can be replicated. As a result, the only case to consider from the Ramsey Problem is an equilibrium of Type 1.

The only difference between the case with a "price" ex-ante instrument and a "quantity" instrument — minimum capital requirement — is that the period zero budget constraints becomes

$$k_0(1 + \tau_{k_0}^i) - n + T_{k_0}^i \leq \sum_s \pi_s d_{1s} \quad [z_0]$$  \hspace{1cm} (78)

where $\tau_{k_0}^i$ is the tax on period zero capital. Also $T_{k_0}^i = -\frac{1}{N} k_0^i T_{k_0}$ is the average transfer of the tax receipts from the government to banker $i$. The banker chooses $k_0^i$ at the end of $t = 0$, while $\tau_{k_0}^i$ is determined in the beginning of $t = 0$ and banker $i$ takes it as given. However banker $i$ internalizes the fact that he affects the transfer $T_{k_0}^i$. The first order condition with respect to $k_0^i$ becomes

$$\max_{k_0^i, d_{1s}^i, k_{1s}^i} \sum_s \pi_s (A + 1 - \theta) k_{1s}^i$$

subject to

$$k_{1s}^i \left( \tilde{F}' (k_{1s}^i) - \theta \right) + d_{1s}^i \leq \left( \tilde{F}' (k_{1s}^i) + a_{1s} - \gamma \right) k_0^i + B_s \quad [\pi_s z_{1s}^i]$$  \hspace{1cm} (79)

$$d_{1s}^i \leq \theta \left( \tilde{F}' (k_{1s}^i) - \gamma \right) k_0^i \quad [\pi_s \lambda_{1s}^i]$$

First order condition with respect to $k_{i,0}$

$$\sum_s \pi_s z_{1s}^i \left( \tilde{F}' (k_{1s}^i) + a_{1s} - \gamma + \frac{1}{N} \tilde{F}'' (k_{1s}^i) k_{0s} + \frac{\partial B_i}{\partial k_0} \right)$$

$$- z_0 \left[ 1 + \tau_{k_0}^i \left( 1 - \frac{1}{N} \right) \right] + \sum_s \pi_s \lambda_{1s}^i \theta \left( \tilde{F}' (k_{1s}^i) - \gamma + \frac{1}{N} \tilde{F}'' (k_{1s}^i) k_0^i \right) = 0$$  \hspace{1cm} (80)

and all the other First order conditions are the same. Using the fact that $z_0 = z_{1l} > z_{1h}$ and $\lambda_{2s} > 0, \lambda_{1l} = 0$ and $\lambda_{1h} > 0$, re-write the first order condition with respect to $k_0$ from the Ramsey Problem.

$$z_{1l}^{CE,N} \left[ 1 - \pi_h \theta (1 - \gamma) + \tau_{k_0}^i \left( 1 - \frac{1}{N} \right) - \pi_t (\theta + a_{1l} - \gamma) \right] = \frac{A + 1 - \theta}{1 - \theta} \left[ \pi_h ((1 - \theta) (1 - \gamma) + a_{1l}) + \pi_t (1 - \theta) \right]$$  \hspace{1cm} (81)

71
where \( z_{CE,N}^{l} \) is given by equation 68. From equation 58

\[
z_{CE,N}^{l} (1 - \pi_h \theta [1 - \gamma] - \pi_l (\theta + a_{1l} - \gamma)) = \frac{A + 1 - \theta}{(1 - \theta)} [\pi_l (1 - \theta) + \pi_h ((1 - \theta) (1 - \gamma) + a_{1h})]
\]

(82)

where \( z_{CP}^{l} \) is given by equation 59. Subtracting equation 81 from equation 82, since \( z_{CP}^{l} \leq 0 \), which I proved in Proposition 7,

\[
\tau_{k_0} = \left[ \frac{z_{CP}^{l}(k_{1l}^{T})}{z_{CE,N}^{l}(k_{1l}^{T}, \chi)} - 1 \right] \Phi > 0
\]

where \( \Phi = \frac{[1 - \pi_h \theta (1 - \gamma) + \pi_l (\gamma - \theta - a_{1l})]}{(1 - \theta)} > 0 \). Since the equilibrium \( k_{1l}^{T} \) does not vary with \( \chi \) in the Central Planner problem and \( z_{CP}^{l} \) is a function only of \( k_{1l}^{T} \), \( \frac{\partial z_{CP}^{l}(k_{1l}^{T})}{\partial \chi} = 0 \)

\[
\frac{\partial \tau_{k_0}}{\partial \chi} = - \frac{\partial z_{CE,N}^{l}(k_{1l}^{T}, \chi)}{\partial \chi} \frac{z_{CP}^{l}(k_{1l}^{T})}{z_{CE,N}^{l}(k_{1l}^{T}, \chi)} \frac{z_{CE,N}^{l}(k_{1l}^{T}, \chi)}{z_{CE,N}^{l}(k_{1l}^{T}, \chi)} \frac{\partial z_{CE,N}^{l}(k_{1l}^{T}, \chi)}{\partial \chi} \leq 0
\]

where

\[
\frac{\partial z_{CE,N}^{l}(k_{1l}^{T}, \chi)}{\partial \chi} = - \frac{\partial B_l}{\partial k_{1l}^{T}} \frac{z_{CE,N}^{l}(k_{1l}^{T}, \chi)}{\partial \chi} \left[ \tilde{F}^{l}(k_{1l}^{T}) - \theta + \frac{1}{N} \tilde{F}^{u}(k_{1l}^{T}) k_{1l}^{T} + \frac{\partial B_l}{\partial \delta_{\gamma}} \right] \geq 0
\]

where in Corollary 1 I proved \( \frac{\partial B_l}{\partial k_{1l}^{T}} \leq 0 \). Also I take into account that the equilibrium \( k_{1l}^{T} \) is not a function of \( \chi \) since it’s determined by the first order condition of the Central Planner.

If \( N \to \infty \), \( z_{CE,N}^{l} \) is not a function of \( \chi \) because \( \frac{\partial B_l}{\partial k_{1l}^{T}} = 0 \) which implies \( \frac{\partial \tau_{k_0}}{\partial \chi} = 0 \). If \( 1 < N < \infty \), \( \frac{\partial B_l}{\partial k_{1l}^{T}} < 0 \) and \( \frac{\partial \tau_{k_0}}{\partial \chi} < 0 \).