

THE ENVIRONMENTAL IMIMPACT AND SUSTAINABILITY APPPLIED GENERAL EQUILIBRIUM (ENVISAGE) MODEL

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NOTE: This paper will be updated in the coming months as the ENVISAGE model has undergone significant upgrades in the last 18 months. The working version of ENVISAGE now includes multiple power-generating activities, all of the Kyoto greenhouse gases, a new climate module (to take into account the additional greenhouse gases), a wide-variety of damage functions that link temperature change to economic impacts and is now calibrated to the GTAP 7.1 database. The existing paper should be cited with the 2008 date.

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Introduction

The purpose of this document is to provide a complete specification of the equations of the World Bank's ENVIRONMENTAL IMPACT AND SUSTAINABILITY APPLIED GENERAL EQUILIBRIUM (ENVISAGE) MODEL. The ENVISAGE Model is designed to analyze a variety of issues related to the economics of climate change:

- Baseline emissions of CO₂ and other greenhouse gases
- Impacts of climate change on the economy
- Adaptation by economic agents to climate change
- Greenhouse gas mitigation policies—taxes, caps and trade
- The role of land use in future emissions and mitigation
- The distributional consequences of climate change impacts, adaptation and mitigation—at both the national and household level.

ENVISAGE is intended to be flexible in terms of its dimensions. The core database—that includes energy volumes and CO₂ emissions—is the GTAP database, currently version 7.0 with a 2004 base year. The latter divides the world into 113 countries and regions, of which 95 are countries and the other region-based aggregations.¹ The database divides global production into 57 sectors—with extensive details for agriculture and food and energy (coal mining, crude oil production, natural gas production, refined oil, electricity, and distributed natural gas). Annex 7 provides more detail. Due to numerical and algorithmic constraints, a typical model is limited to some 20-30 sectors and 20-30 regions.

This document describes the current version of ENVISAGE, which is still in a developmental stage. This current version includes the following:

- Capital vintage production technology that permits analysis of the flexibility of economies
- A detailed specification of energy demand in each economy, with additions yet to come (see below)
- The ability to introduce future alternative energy (or backstop) technologies
- CO₂ emissions that are fuel and demand specific
- A flexible system for incorporating any combination of carbon taxes, emission caps and tradable permits
- A simplified climate module that links greenhouse gas emissions to atmospheric concentrations combined with a carbon cycle that leads to radiative forcing and temperature changes.

The future work program includes the following tasks:

- Splitting electricity into nuclear, hydro, renewables and other
- Adding a resource depletion module for coal, oil and gas

¹ The countries defined in GTAP cover well over 90 percent of global GDP and population. The country coverage is weakest for Sub-Saharan Africa and the Middle East—though with ongoing work to extend the country coverage.

- Adding other greenhouse gases (for example those linked to agriculture)
- Adding a more detailed land-use module
- Adding additional alternative technologies

Model specification

Table 1: Sets used in model definition

<i>Set</i>	<i>Description</i>
<i>aa</i>	Armington agents
<i>a</i>	Activities (a subset of <i>aa</i>)
<i>i</i>	Produced goods
<i>Manu</i>	Set of manufacturing sectors (subset of <i>i</i> , used in definition of numéraire)
<i>fp</i>	Factors of production
<i>l</i>	Labor categories (subset of <i>fp</i>)
' <i>Captl</i> '	Capital account (subset of <i>fp</i>)
' <i>Landr</i> '	Land account (subset of <i>fp</i>)
' <i>Natrs</i> '	Natural resource account (subset of <i>fp</i>)
<i>in</i>	Institutions (subset of <i>a</i>)
<i>h</i>	Households (subset of <i>in</i>)
<i>Gov</i>	Government account (subset of <i>in</i>)
<i>Inv</i>	Investment account (subset of <i>in</i>)
<i>gy</i>	Government revenue accounts
' <i>ptax</i> '	Production tax account (subset of <i>gy</i>)
' <i>dtax</i> '	Sales tax account on domestically produced goods sold domestically (subset of <i>gy</i>)
' <i>mtax</i> '	Sales tax account on imported goods (subset of <i>gy</i>)
' <i>ttax</i> '	Import tariff account (subset of <i>gy</i>)
' <i>etax</i> '	Export tax account (subset of <i>gy</i>)
' <i>vtax</i> '	Tax on factors of production account (subset of <i>gy</i>)
<i>r</i>	Regions
<i>r'</i>	Alias with <i>r</i>
<i>HIC</i>	Set of high-income regions (subset of <i>r</i> , used in definition of numéraire)
<i>RSAV</i>	Residual region (subset of <i>r</i> , must be of single dimension)

Production block

The ENVISAGE production structure relies on a set of nested constant-elasticity-of-substitution (CES) structures²—it has somewhat less flexibility than that developed for the Linkage model, but somewhat more than in the standard GTAP model. Like Linkage, production, in the dynamic version of the model is based on a vintage structure of capital, indexed by v . In the standard version, there are two vintages—*Old* and *New*, where *New* is capital equipment that is newly installed at the beginning of the period and *Old* capital is capital greater than a year old. The vintage structure impacts model results through two channels. First, it is typically assumed that *Old* capital has lower substitution elasticities than *New* capital. Thus countries with higher savings rates will have a higher share of *New* capital and thus greater overall flexibility. The

² Some of the key analytical properties of the CES, and its related constant-elasticity-of-transformation (CET) function, are fully described in Annex 1.

second channel is through the allocation of capital across sectors. *New* capital is assumed to be perfectly mobile across sectors. *Old* capital is sluggish and released using an upward sloping supply curve. In sectors where demand is declining, the return to capital will be less than the economy-wide average. This is explained in greater detail in the market equilibrium section.

Most of the equations in the production structure are indexed by v , i.e. the capital vintage. The exceptions are those where it is assumed that the further decomposition of a bundle are no longer vintage specific—such as the demand for non-energy intermediate inputs. Each production activity is indexed by a , and is different from the index of produced commodities, i (allowing for the combination of outputs from different activities into a single produced good, for example electricity).

$$(P-1) \quad VA_{r,a,v} = \alpha_{r,a,v}^{va} \left(\delta_{r,a}^{cd} \lambda_{r,a,v}^v \right)^{\sigma_{r,a,v}^p - 1} \left(\frac{PXv_{r,a,v}}{PVA_{r,a,v}} \right)^{\sigma_{r,a,v}^p} XPv_{r,a,v}$$

$$(P-2) \quad ND_{r,a} = \sum_v \alpha_{r,a,v}^{nd} \left(\delta_{r,a}^{cd} \lambda_{r,a,v}^n \right)^{\sigma_{r,a,v}^p - 1} \left(\frac{PXv_{r,a,v}}{PND_{r,a}} \right)^{\sigma_{r,a,v}^p} XPv_{r,a,v}$$

$$(P-3) \quad PXv_{r,a,v} = \frac{1}{\delta_{r,a}^{cd}} \left[\alpha_{r,a,v}^{va} \left(\frac{PVA_{r,a,v}}{\lambda_{r,a,v}^v} \right)^{1 - \sigma_{r,a,v}^p} + \alpha_{r,a,v}^{nd} \left(\frac{PND_{r,a}}{\lambda_{r,a,v}^n} \right)^{1 - \sigma_{r,a,v}^p} \right]^{1/(1 - \sigma_{r,a,v}^p)}$$

$$(P-4) \quad PX_{r,a} = (1 + \pi_{r,a}) \frac{\sum_v PXv_{r,a,v} XPv_{r,a,v}}{XP_{r,a}}$$

$$(P-5) \quad \Pi_{r,a} = \pi_{r,a} \sum_v PXv_{r,a,v} XPv_{r,a,v}$$

$$(P-6) \quad PP_{r,a} = (1 + \tau_{r,a}^p) PX_{r,a} + \tau_{r,a}^x$$

Equations (P-1) and (P-2) are derived demands for two bundles, one designated as aggregate value added, VA , though it also includes energy demand that is linked to capital, and aggregate intermediate demand, ND , a bundle that excludes energy. Both are shares of output by vintage, XPv , with the shares being price sensitive with respect to the ratio of the vintage-specific unit cost, PXv , and the component prices, respectively PVA and PND . The equations allow for technological change embodied in the λ parameters that are allowed to be node-specific. For uniform technological change, the two parameters can be subject to the same percentage change. Both productivity factors are impacted by the same damage adjustment, δ^{cd} , which is region and sector specific and depends on climate change.³ Equation (P-3) defines the vintage-specific unit cost, PXv . Almost all CES price equations are based on the dual cost function instead of the aggregate cost or revenue formulation. The unit cost function includes the effects of productivity

³ Discussed further below.

improvement and damages. To the extent climate leads to damages, δ^{cd} drops below its initial level of 1, raising unit cost, all else equal. Equation (P-4) determines the aggregate unit cost, PX , the weighted average of the vintage-specific unit costs with the weights given by the vintage-specific output levels. The model allows for a markup, π , to unit cost that is normally exogenous and initialized at 0. The revenue generated by the markup, Π , is defined in equation (P-5). Equation (P-6) determines the final market price for output, PP , that is equal to the unit cost augmented by the output tax (or subsidy), τ^p . The equivalence of the tax-adjusted unit cost to the output price is an implication of assuming constant-returns-to-scale technology and perfect competition (and/or the presence of a fixed markup). The production price can also be adjusted by a volume only tax (or an excise tax), represented by τ^x .

The subsequent production nest decomposes the VA bundle (value added and energy) into non-capital factors of production on the one hand, XF , indexed by fpz^4 , and the capital/energy bundle on the other hand, KE . The key substitution elasticity is given by σ^v . An elasticity of 1 implies a Cobb-Douglas technology.⁵ Equation (P-7) determines the demand for non-capital factors (unskilled and skilled labor, land, and a sector-specific factor if it exists).⁶ Factor productivity is given by the λ factor. Equation (P-8) determines demand for the capital/energy bundle, KE . The final equation in this nest (P-9) defines the unit price of the value added cum energy bundle, PVA .

$$(P-7) \quad XF_{r,fpz,a}^d = \sum_v \alpha_{r,fpz,a,v}^f \left(\lambda_{r,fpz,a}^f \right)^{\sigma_{r,a,v}^v - 1} \left(\frac{PVA_{r,a,v}}{PF_{r,fpz,a}} \right)^{\sigma_{r,a,v}^v} VA_{r,a,v}$$

$$(P-8) \quad KE_{r,a,v} = \alpha_{r,a,v}^{ke} \left(\frac{PVA_{r,a,v}}{PKE_{r,a,v}} \right)^{\sigma_{r,a,v}^v} VA_{r,a,v}$$

$$(P-9) \quad PVA_{r,a,v} = \left[\sum_{fpz} \alpha_{r,fpz,a,v}^f \left(\frac{PF_{r,fpz,a}}{\lambda_{r,fpz,a}^f} \right)^{1 - \sigma_{r,a,v}^v} + \alpha_{r,a,v}^{ke} PKE_{r,a,v}^{1 - \sigma_{r,a,v}^v} \right]^{1/(1 - \sigma_{r,a,v}^v)}$$

The next nest is a decomposition of the capital/energy bundle, KE , into demand for capital (by vintage) and an energy bundle. Equation (P-10) defines the demand for capital by vintage, KV . The substitution elasticity is given by σ^{ke} . Equation (P-11) determines the demand for the energy bundle, $XNRG$. The latter is indexed by eb , a special set that indexes all energy bundles. There is a set mapping that has a one-to-one correspondence between the given activity a and a specific item in eb . The reason for this is to simplify the code for disaggregating the energy bundles

⁴ The set fp indexes all factors of production, the subset fpz excludes capital.

⁵ In the GAMS implementation of the model, a Cobb-Douglas technology is approximated by an elasticity of 1.01.

⁶ The model implementation allows for a scale factor (**phiw**) that is used to scale factor prices. This can help with numerical problems.

across agents in the economy and is described further below. Equation (P-12) defines the price of the KE bundle, PKE .

$$(P-10) \quad KV_{r,a,v} = \alpha_{r,CapIt,a,v}^f \left(\lambda_{r,CapIt,a,v}^f \right)^{\sigma_{r,a,v}^{ke} - 1} \left(\frac{PKE_{r,a,v}}{PKV_{r,a,v}} \right)^{\sigma_{r,a,v}^{ke}} KE_{r,a,v}$$

$$(P-11) \quad XNRG_{r,eb,v} = \alpha_{r,a,v}^{ep} \left(\frac{PKE_{r,a,v}}{PNRG_{r,eb,v}} \right)^{\sigma_{r,a,v}^{ke}} KE_{r,a,v}$$

$$(P-12) \quad PKE_{r,a,v} = \left[\alpha_{r,CapIt,a,v}^f \left(\frac{PKV_{r,a,v}}{\lambda_{r,CapIt,a}^f} \right)^{1 - \sigma_{r,a,v}^{ke}} + \alpha_{r,a,v}^{ep} PNRG_{r,eb,v}^{1 - \sigma_{r,a,v}^{ke}} \right]^{1/(1 - \sigma_{r,a,v}^{ke})}$$

The final node in the production nest is the decomposition of aggregate demand for non-fuel intermediate goods, ND . For the moment, we are assuming a standard Leontief technology (although allowing for the possibility of substitution across inputs.⁷ Equation (P-13) determines the demand for the (Armington) intermediate demand for non-fuel inputs, XA_n , with the substitution elasticity given by σ^n . The relevant price is the agent (or activity) specific Armington price, PA^a . The latter will be a composite price of domestic and imported goods, augmented by domestic taxes and, in mitigation scenarios, with a tax linked to emissions (described below). The model allows for input-specific efficiency improvements as encapsulated by the λ^{nd} parameter. Equation (P-14) provides the price of the aggregate ND bundle.

$$(P-13) \quad XA_{r,n,a} = iO_{r,n,a,Old} \left(\lambda_{r,n,a}^{nd} \right)^{\sigma_{r,a}^n - 1} \left(\frac{PND_{r,a}}{PA_{r,n,a}^a} \right)^{\sigma_{r,a}^n} ND_{r,a}$$

$$(P-14) \quad PND_{r,a} = \left[\sum_{n \in nrg} iO_{r,n,a,Old} \left(\frac{PA_{r,n,a}^a}{\lambda_{r,n,a}^{nd}} \right)^{1 - \sigma_{r,a}^n} \right]^{1/(1 - \sigma_{r,a}^n)}$$

This ends the description of the production structure, though there is a further decomposition of XA_n , i.e. the non-fuels intermediate Armington demand, and the energy bundle. The latter is decomposed across fuels, and then finally decomposed as an Armington good.

Income block

⁷ The Linkage model has a different production structure for crops, livestock and all other goods allowing for more complex interactions between agricultural inputs, for example fertilizers and feed, and the factors of production.

The model has six indirect tax streams and one direct-tax stream:

1. The output tax, τ^p imposed on the aggregate price of output, PX , with an additional excise tax, τ^x , in some circumstances.
2. A sales tax on sales of domestic Armington goods, τ^{Ap} , which is agent specific and imposed on the economy-wide price of domestic goods, PA .⁸
3. Bilateral import tariff, τ^m , imposed on the landed (or CIF) price of imports, WPM . The model also allows for homogeneous goods, in which case the tariff represents a wedge between the world price and the domestic price.
4. Bilateral export tax (or subsidy), τ^e , imposed on the producer price of exports, PE . In the case of a homogeneous commodity, the export tax represents the wedge between world prices and domestic prices.
5. Taxes on the factors of production, τ^v , imposed on the market-clearing price of factors, NPF .
6. Taxes on emissions, τ^e , imposed on the Armington consumption of goods.

Equations (Y-1) through (Y-6) correspond to the aggregate revenues generated by each of the six indirect taxes. The notation for the variables not already described will be given below. One important observation concerns the bilateral trade variables. These are always indexed as (r, r', i) where r is the country of origin (the exporter), r' the country of destination (the importer) and i is the sector index. This explains the switch in the indices equations (Y-3) and (Y-4) where WTF corresponds to the bilateral trade flow from region r to region r' . Equation (Y-7) defines the aggregate revenue from taxing household income. Fiscal closure will be discussed below.

$$(Y-1) \quad GREV_{r,ptax} = \sum_a \left[\tau_{r,a}^p PX_{r,a} XP_{r,a} + \tau_{r,a}^x XP_{r,a} \right]$$

$$(Y-2) \quad GREV_{r,atax} = \sum_i \sum_{aa} \tau_{r,i,aa}^{Ap} (\gamma_{r,i,aa}^c PAT_{r,i}) XA_{r,i,aa}$$

$$(Y-3) \quad GREV_{r,ttax} = \sum_{i \in Arm} \sum_{r'} \tau_{r',r,i}^m WPM_{r',r,i} WTF_{r',r,i}^d + \sum_{i \notin Arm} \tau_{r,i}^m PW_i XMT_{r,i}$$

$$(Y-4) \quad GREV_{r,etax} = \sum_{i \in Arm} \sum_{r'} \tau_{r,r',i}^e PE_{r,r',i} WTF_{r,r',i}^s + \sum_{i \notin Arm} \tau_{r,i}^e PW_i XET_{r,i}$$

$$(Y-5) \quad GREV_{r,vtax} = \sum_{fp} \sum_a \tau_{r,fp,a}^v NPF_{r,fp,a} XF_{r,fp,a}$$

$$(Y-6) \quad GREV_{r,ctax} = \sum_{em} \sum_i \sum_{aa} \tau_{r,em}^{emi} \phi_{r,em,i,aa} \rho_{r,em,i,aa} XA_{r,i,aa}$$

$$(Y-7) \quad GREV_{r,htax} = \sum \chi_r^k \kappa_{r,h}^h YH_r$$

⁸ The model allows for agent-specific Armington decompositions, though this increases the size of the model considerably. This is described further in Annex 3 on alternative trade specification.

Equation (Y-8) defines aggregate fiscal revenues, where the set gy corresponds to the six indirect tax streams ($ptax$, $atax$, $ttax$, $etax$, $vtax$ and $ctax$) and the direct tax stream ($htax$). It is also assumed that income from a cap and trade system on emissions accrue to the government. Equation (Y-9) summarizes net household income, YH . It is assumed that all factor income net of factor taxes (where NPF represents the market clearing factor price net of taxes) accrues to households as well as profits generated by the markups. Household income is then adjusted for the depreciation allowance, $DeprY$. Equation (Y-10) describes household disposable income, YD , where κ^h represents the base year household-specific direct tax rate. The direct tax rate is adjusted by an economy-wide adjustment factor, χ^k , which can be endogenous to achieve a given target, for example the deficit of the public sector. Macro closure is discussed in more detail below.⁹

$$(Y-8) \quad YG_r = \sum_{gy} GREV_{r,gy} + \sum_{em} QuotaY_{em}^E$$

$$(Y-9) \quad YH_r = \sum_{fp} \sum_a NPF_{r,fp,a} XF_{r,fp,a} + \sum_a \Pi_{r,a} - DeprY_r$$

$$(Y-10) \quad YD_r = (1 - \chi_r^k \kappa_{r,h}^h) YH_r$$

Demand block

The demand block is divided into two sections. The first describes the allocation of household disposable income between savings and expenditures on goods and services. The second describes other final demand for goods and services.

Households first allocate total expenditures between savings on the one hand and aggregate expenditures on goods and services on the other hand.¹⁰ Equation (D-1) determines the household savings rate (relative to disposable income), s^s , as a function of per capita growth (g^{pc}) and the youth and elderly dependency ratios, respectively given by $DRAT^{PLT15}$ and $DRAT^{P65UP}$.¹¹ These variables are typically exogenous in dynamic scenarios. The savings function also

⁹ The GTAP dataset contains only a single representative household per country/region. The model implementation allows for multiple households and hence the need for an economy-wide tax shifter that is uniform across households. This has different distributional consequences than an additive shifter or a more complex direct tax schedule.

¹⁰ The demand block is significantly reformulated compared to the first version of the ENVISAGE model. The latter was largely inspired by the GTAP model. The current version is more similar to the Linkage specification. It drops the top level utility function that allocated national income across savings, and public and private expenditures. In the long-term scenarios this top-level structure was typically over-ruled with other specific assumptions making the theoretical consistency of the top-level formulation less appealing.

¹¹ The original theory and parameters for this formulation can be found in Loayza et al 2000 and Masson et al 1998 and is summarized in van der Mensbrugge 2006.

captures a persistence factor defined by β^s .¹² Equation (D-2) determines the level of household savings. It should be noted that if the ELES utility function is used to specify household demand for goods and services, equation (D-1) is dropped and equation (D-2) then defines the average propensity to save as the ELES itself determines the level of savings. Equation (D-3) in essence determines aggregate expenditures on household goods and services, YC . In the case of the ELES, combined with equation (D-4), it defines the level of household savings, which is an outcome of the ELES. Equation (D-4) is only used for the ELES version of the model.

$$(D-1) \quad s_{r,h}^s = \chi_r^s \alpha_{r,h}^s + \beta_r^s s_{r,h,-1}^s + \beta_r^g g_r^{pc} + \beta_r^y DRAT_r^{PLT15} + \beta_r^e DRAT_r^{P65UP}$$

$$(D-2) \quad S_{r,h}^h = s_{r,h}^s YD_{r,h}$$

$$(D-3) \quad YD_{r,h} = YC_{r,h} + S_{r,h}^h$$

$$(D-4) \quad YC_{r,h} = \sum_k PHX_{r,k,h} XH_{r,k,h}$$

Expenditures

The next block of equations determines the sectoral demands for goods and services for households. In the standard model private expenditures are derived from the CDE specification and are based on consumer-defined goods, XH , not Armington goods, XA .¹³ Equation (D-5) represents the (implicit) utility function for the CDE, where PHX are consumer prices. Equation (D-6) reflects the first order conditions. Demand is specified on a per capita basis so total private expenditure is divided by population to define per capita expenditure and per capita demand is then re-multiplied by population to get total private demand.¹⁴ Equation (D-7) represents the private expenditure budget shares, s^h . Equation (D-8) defines a consumer price index for private consumption, PC . Finally, equation (D-9) defines the volume of aggregate private consumption, XC .

¹² Setting all the β parameters to 0 would yield a constant savings rate. It is also possible to use this equation to formulate a different closure—for example to target investment and allow the shift parameter, χ^s , to adjust to achieve the given target.

¹³ Other demand specifications have been implemented and are described in Annex x. These include both the linear and extended linear expenditure system (LES and ELES) as well as the AIDADS specification that is a natural extension of the LES. Three of the expenditure systems (CDE, LES and AIDADS) use a two-tiered nest to allocate savings on the one hand and expenditures on goods and services on the other. The ELES integrates both in a single-tiered system.

¹⁴ Equation (D-6) is split into two in the GAMS code with the denominator calculated in a separate expression.

$$(D-5) \quad \sum_k \alpha_{r,k,h}^h U_{r,h}^{e_{r,k,h}^h b_{r,k,h}^h} \left(\frac{PHX_{r,k,h}}{(YC_{r,h} / Pop_{r,h})} \right)^{b_{r,k,h}^h} \equiv 1$$

$$(D-6) \quad HX_{r,k,h} = Pop_{r,h} \frac{\alpha_{r,k,h}^h b_{r,k,h}^h U_{r,h}^{e_{r,k,h}^h b_{r,k,h}^h} \left(\frac{PHX_{r,k,h}}{(YC_{r,h} / Pop_{r,h})} \right)^{b_{r,k,h}^h - 1}}{\sum_{k'} \alpha_{r,k',h}^h b_{r,k'}^h U_{r,h}^{e_{r,k',h}^h b_{r,k'}^h} \left(\frac{PHX_{r,k',h}}{(YC_{r,h} / Pop_{r,h})} \right)^{b_{r,k',h}^h}}$$

$$(D-7) \quad s_{r,k,h}^h = \frac{PHX_{r,k,h} HX_{r,k,h}}{YC_{r,h}}$$

$$(D-8) \quad PC_{r,h} = \sum_k s_{r,k,h}^h PHX_{r,k,h}$$

$$(D-9) \quad XC_{r,h} = YC_{r,h} / PC_{r,h}$$

The next set of equations decomposes consumer demand defined as consumer goods into produced goods. A transition matrix approach is used where each consumed good is composed of one or more produced goods and combined using a CES aggregator.¹⁵ Each consumer good can also have its own energy bundle—with different demand shares across energy.¹⁶

$$(D-10) \quad XA_{r,n,h} = \sum_k \alpha_{r,n,k,h}^c \left(\frac{PHX_{r,k,h}}{PA_{r,n,h}^a} \right)^{\sigma_{r,k,h}^c} HX_{r,k,h}$$

$$(D-11) \quad XNRG_{r,eb,Old} = \sum_h \alpha_{r,k,h}^{nrgh} \left(\frac{PHX_{r,k,h}}{PNRG_{r,eb,Old}} \right)^{\sigma_{r,k,h}^c} HX_{r,k,h}$$

$$(D-12) \quad PHX_{r,k,h} = \left[\sum_{n \in nnrgh} \alpha_{r,n,k,h} (PA_{r,n,h}^a)^{1-\sigma_{r,k,h}^c} + \alpha_{r,k,h}^{nrgh} (PNRG_{r,eb,Old})^{1-\sigma_{r,k,h}^c} \right]^{1/(1-\sigma_{r,k,h}^c)}$$

The final block of demand equations decomposes aggregate public and investment demands. A CES expenditure function is used that covers all non-energy Armington goods and an energy

¹⁵ Using the standard GTAP data, the transition matrix is diagonal—each consumed good corresponds to exactly one produced good. ENVISAGE still uses this approach save for the energy bundle that is combined into one consumed commodity. Work is ongoing to develop a global database of transition matrices. The GREEN model for example (see Burniaux et al (199x) and van der Mensbrugge (199x)) had four consumed goods and eight standard produced goods.

¹⁶ For example, a transportation bundle is likely to be dominated by liquid fuel demand, whereas demand for heat is likely to be dominated by electricity and natural gas.

bundle. Decomposition of the energy bundle is done at a latter stage. Equation (D-13) represents the sectoral (Armington) demand for public and investment non-energy expenditures XA , where the index f represents the set spanning (*gov* and *inv*). Equation (D-14) determines the demand for the energy bundle (where the index *eb* is mapped to the respective f index). The expenditure price indices, PC_f , are given by equation (D-15). In the standard model there are no stock-building activities. In some scenarios it is helpful to give ‘exogenous’ demand shocks. This is most easily done by assuming stock-building activities as defined in equation (D-16), where the level of stock-building is linked to domestic production, XS .

$$(D-13) \quad XA_{r,n,f} = \alpha_{r,n,f}^f \left(\frac{PC_{r,f}}{PA_{r,n,f}^a} \right)^{\sigma_{r,f}^f} XC_{r,f}$$

$$(D-14) \quad XNRG_{r,eb,Old} = \alpha_{r,f}^{nrgr} \left(\frac{PC_{r,f}}{XNRG_{r,eb,Old}} \right)^{\sigma_{r,f}^f} XC_{r,f}$$

$$(D-15) \quad PC_{r,f} = \left[\sum_n PA_{r,n,f}^a XA_{r,n,f} + PNRG_{r,eb,Old} XNRG_{r,eb,Old} \right] / XC_{r,f}$$

$$(D-16) \quad XA_{r,i,spb} = \chi_{r,i}^{spb} XS_{r,i}$$

The fuels block

Each agent in the economy has a specified demand for an aggregate energy bundle. The fuel demanders are indexed by *eb* that spans all activities (*a*), each commodity consumed by households (*k*) and other final demand (*f*).¹⁷ The equations above provide the bundle $XNRG$ across all *eb* agents. That bundle is decomposed across all energy sources using a nested CES structure with agent-specific share parameters and substitution elasticities.¹⁸ At the top level, demand is decomposed between electricity and non-electric energy (see figure xx). The non-electric bundle is split into coal on the one hand, and gas and oil on the other. The oil and gas bundle is then split into oil on the one hand and gas on the other. Using the standard GTAP classification, the final electric bundle is composed of commodity *ely* alone. The coal bundle is composed of the commodity *coa* alone. The gas bundle is composed of the commodities *gas* and *gdt*. And the oil bundle is composed of the commodities *oil* and *p_c*. In most cases, for these latter two bundles, one component will dominate the other. For example, there may be some residual *oil* consumption in households, but the bulk of the consumption will be *p_c*. When the new alternative technologies are introduced, they are inserted at the bottom most node for electricity, coal, oil and gas respectively. [To be developed further.]

¹⁷ With the simplified consumer transition matrix, only one consumed commodity demands energy, and that is the entire expenditure on energy in the single consumer demand vector in the existing GTAP data.

¹⁸ Note that in the case of energy bundles from the production side, they are also indexed by vintage with potentially different substitution elasticities across vintages. The non-production energy bundles are also indexed by vintage (for simplicity), though only the *Old* vintage is active.

The next block of equations is the top of the energy node nest. It decomposes the energy bundle, $XNRG$, into an electric bundle, $XELY$, and a non-electric bundle, $XNELY$. Equations (F-1) and (F-2) define respectively the demands for the electric and non-electric bundles with a substitution elasticity given by σ^e . The equations are defined over all demanders of energy bundles, eb , and are also vintage-specific in production. Equation (F-3) defines the CES price of the energy bundle as a CES aggregation of the respective bundle prices, $PELY$ and $PNELY$.

$$(F-1) \quad XELY_{r,eb,v} = \alpha_{r,eb,v}^{ely} \left(\frac{PNRG_{r,eb,v}}{PELY_{r,eb,v}} \right)^{\sigma_{r,eb,v}^e} XNRG_{r,eb,v}$$

$$(F-2) \quad XNELY_{r,eb,v} = \alpha_{r,eb,v}^{nely} \left(\frac{PNRG_{r,eb,v}}{PNELY_{r,eb,v}} \right)^{\sigma_{r,eb,v}^e} XNRG_{r,eb,v}$$

$$(F-3) \quad PNRG_{r,eb,v} = \left[\alpha_{r,eb,v}^{ely} (PELY_{r,eb,v})^{1-\sigma_{r,eb,v}^e} + \alpha_{r,eb,v}^{nely} (PNELY_{r,eb,v})^{1-\sigma_{r,eb,v}^e} \right]^{1/(1-\sigma_{r,eb,v}^e)}$$

The following block decomposes the non-electric bundle into a coal bundle, $XCOA$, and an oil and gas bundle, $XOLG$, given respectively by equations (F-4) and (F-5) with a substitution elasticity of σ^{nely} . Equation (F-6) defines the price of the non-electric bundle, P^{NELY} .

$$(F-4) \quad XCOA_{r,eb,v} = \alpha_{r,eb,v}^{coa} \left(\frac{PNELY_{r,eb,v}}{PCOA_{r,eb,v}} \right)^{\sigma_{r,eb,v}^{nely}} XNELY_{r,eb,v}$$

$$(F-5) \quad XOLG_{r,eb,v} = \alpha_{r,eb,v}^{olg} \left(\frac{PNELY_{r,eb,v}}{POLG_{r,eb,v}} \right)^{\sigma_{r,eb,v}^{nely}} XNELY_{r,eb,v}$$

$$(F-6) \quad PNELY_{r,eb,v} = \left[\alpha_{r,eb,v}^{coa} (PCOA_{r,eb,v})^{1-\sigma_{r,eb,v}^{nely}} + \alpha_{r,eb,v}^{olg} (POLG_{r,eb,v})^{1-\sigma_{r,eb,v}^{nely}} \right]^{1/(1-\sigma_{r,eb,v}^{nely})}$$

The third node decomposes the oil and gas bundle into a gas bundle, $XGAS$, and an oil bundle, $XOIL$. Equations (F-7) and (F-8) provide the demand equations for the respective bundles with a substitution elasticity of σ^{olg} . Finally, equation (F-9) describes the price of the oil and gas bundle, $POLG$, as a CES aggregation of the gas bundle, $PGAS$, and the oil bundle, $POIL$.

$$(F-7) \quad XGAS_{r,eb,v} = \alpha_{r,eb,v}^{gas} \left(\frac{POLG_{r,eb,v}}{PGAS_{r,eb,v}} \right)^{\sigma_{r,eb,v}^{olg}} XOLG_{r,eb,v}$$

$$(F-8) \quad XOIL_{r,eb,v} = \alpha_{r,eb,v}^{oil} \left(\frac{POLG_{r,eb,v}}{POIL_{r,eb,v}} \right)^{\sigma_{r,eb,v}^{olg}} XOLG_{r,eb,v}$$

$$(F-9) \quad POLG_{r,eb,v} = \left[\alpha_{r,eb,v}^{gas} (PGAS_{r,eb,v})^{1-\sigma_{r,eb,v}^{olg}} + \alpha_{r,eb,v}^{oil} (POIL_{r,eb,v})^{1-\sigma_{r,eb,v}^{olg}} \right]^{1/(1-\sigma_{r,eb,v}^{olg})}$$

At this point, the decomposition of fuels is down to four fundamental energy sources—electricity, coal, gas and oil. In the initial state, with the GTAP data alone, each of the six energies in GTAP is mapped to these four bundles. Four energy sets are defined: *ely*, *coa*, *oil* and *gas* that correspond to a mapping to one of the four types of energy. The GTAP *ely* sector is mapped to *ely*, the GTAP *coa* sector is mapped to *coa*, the GTAP *gas* and *gdt* sectors are mapped to *gas*, and the GTAP *oil* and *p_c* sectors are mapped to *oil*. With the introduction of new technologies, the set mappings will increase. Thus if there is one electric backstop technology, say renewables, and designated by *elybs*, it will be mapped to the *ely* aggregate electric bundle.

$$(F-10) \quad XA_{r,ely,aa} = \sum_{eb \in aa} \sum_v \alpha_{r,ely,eb,v}^{elybs} (\lambda_{r,ely,eb,v}^e)^{\sigma_{r,eb,v}^{ely}} - 1 \left(\frac{PELY_{r,eb,v}}{PA_{r,ely,aa}^a} \right)^{\sigma_{r,eb,v}^{ely}} XELY_{r,eb,v}$$

$$(F-11) \quad PELY_{r,eb,v} = \left[\sum_{aa \in eb} \sum_{ely} \alpha_{r,ely,eb,v}^{elybs} \left(\frac{PA_{r,ely,aa}^a}{\lambda_{r,ely,eb,v}^e} \right)^{1 - \sigma_{r,eb,v}^{ely}} \right]^{1/(1 - \sigma_{r,eb,v}^{ely})}$$

$$(F-12) \quad XA_{r,coa,aa} = \sum_{eb \in aa} \sum_v \alpha_{r,coa,eb,v}^{coabs} (\lambda_{r,coa,eb,v}^e)^{\sigma_{r,eb,v}^{coa}} - 1 \left(\frac{PCOA_{r,eb,v}}{PA_{r,coa,aa}^a} \right)^{\sigma_{r,eb,v}^{coa}} XCOA_{r,eb,v}$$

$$(F-13) \quad PCOA_{r,eb,v} = \left[\sum_{aa \in eb} \sum_{coa} \alpha_{r,coa,eb,v}^{coabs} \left(\frac{PA_{r,coa,aa}^a}{\lambda_{r,coa,eb,v}^e} \right)^{1 - \sigma_{r,eb,v}^{coa}} \right]^{1/(1 - \sigma_{r,eb,v}^{coa})}$$

$$(F-14) \quad XA_{r,gas,aa} = \sum_{eb \in aa} \sum_v \alpha_{r,gas,eb,v}^{gasbs} (\lambda_{r,gas,eb,v}^e)^{\sigma_{r,eb,v}^{gas}} - 1 \left(\frac{PGAS_{r,eb,v}}{PA_{r,gas,aa}^a} \right)^{\sigma_{r,eb,v}^{gas}} XGAS_{r,eb,v}$$

$$(F-15) \quad PGAS_{r,eb,v} = \left[\sum_{aa \in eb} \sum_{gas} \alpha_{r,gas,eb,v}^{gasbs} \left(\frac{PA_{r,gas,aa}^a}{\lambda_{r,gas,eb,v}^e} \right)^{1 - \sigma_{r,eb,v}^{gas}} \right]^{1/(1 - \sigma_{r,eb,v}^{gas})}$$

$$(F-16) \quad XA_{r,oil,aa} = \sum_{eb \in aa} \sum_v \alpha_{r,oil,eb,v}^{oilbs} (\lambda_{r,oil,eb,v}^e)^{\sigma_{r,eb,v}^{oil}} - 1 \left(\frac{POIL_{r,eb,v}}{PA_{r,oil,aa}^a} \right)^{\sigma_{r,eb,v}^{oil}} XOIL_{r,eb,v}$$

$$(F-17) \quad POIL_{r,eb,v} = \left[\sum_{aa \in eb} \sum_{oil} \alpha_{r,oil,eb,v}^{oilbs} \left(\frac{PA_{r,oil,aa}^a}{\lambda_{r,oil,eb,v}^e} \right)^{1 - \sigma_{r,eb,v}^{oil}} \right]^{1/(1 - \sigma_{r,eb,v}^{oil})}$$

Equations (F-10) through (F-17) determine the decomposition of the four basic energy bundles to their respective Armington volumes. For electricity and coal, with the base data, these equations are somewhat redundant since the bundles map to only one Armington commodity. Each demand equation requires a summing over vintages (for only activities), and a summing across eb indices. In most cases, the eb index maps to one, and only one, agent (aa). In the case of consumption, however, the energy bundle can exist for each consumed commodity (k), and thus there can be as many energy decompositions as there are consumer commodities. Each bundle also allows for energy efficiency improvement, sometimes designated as the autonomous energy efficiency improvement (AEEI) parameter, which is region, agent, fuel and vintage specific (in principle). The price equations need a separate mapping from the eb to the aa index, though it is assumed that the consumer price for a given fuel is uniform across the k commodities (i.e. natural gas used for heat has the same price as natural gas used for transportation.).

Trade block

Top level Armington

The equations above have determined completely the so-called Armington demand for goods across all agents, XA , that include activities (a), private or consumer demand (h), and other final demand (f). The union of these three sets is the set aa . In the standard version of ENVISAGE, all Armington agents are assumed to have the same preference function for domestic and import goods.¹⁹ It is also assumed that the Armington good, for each commodity i , is homogeneous across agents, and can therefore be aggregated in volume terms. However, when using the energy volume data that comes with the GTAP data set, the derived energy prices vary (modestly in most cases) across agents.²⁰ To maintain the adding up assumption with the price differentials, a shift parameter is associated with each agent. One could think of this intuitively as a quality index, so the gasoline consumed by households has a different quality than that consumed in transportation, where quality differences may simply reflect octane levels.

Equation (T-1) defines aggregate Armington demand, XAT . It is the sum across all agents of their Armington demand—adjusted by the fixed shift (or quality) parameter, γ^c .²¹ The agent-specific Armington price is composed of two components. The first, PA , is formed from the nationally determined Armington price, PAT , defined below, adjusted by the quality index, γ^c , and augmented by the user-specific sales tax, τ^{Ap} —see equation (T-2). To this is added the emission tax, τ^{emi} , see equation (T-3). The emissions tax is given as a \$ amount per unit of emission, where ρ determines the agent specific level of emissions per unit of demand by agent (aa), per input (i) and per emission type (em). The model allows for full or partial exemptions using the parameter ϕ —that can also be agent, input and emission specific. For example it is possible to exempt given sectors or households from paying the emissions tax for specific fuels, say gasoline. By default, the parameter ϕ is set at 1, i.e. there are no exemptions. The emissions tax can be either set exogenously or be model-derived by imposing an emissions cap at either the country or regional level.²² Notice that the emission tax is not an ad valorem tax, but a Pigouvian per unit tax.

¹⁹ The GTAP data decomposes Armington demand into its domestic and import component by agent. Annex 3 explains an alternative version of the Armington decomposition that allows for agent-specific behavior. Note that this increases the size of the model considerably.

²⁰ The energy data, derived from the databases of the International Energy Agency (IEA), are expressed in millions of tons of oil equivalent (MTOE) across all energies, and thus prices are \$2004 prices per unit of MTOE.

²¹ The γ^c parameter is initialized at 1 for all non-energy commodities. For energy commodities, it is initialized such that there is uniformity of energy prices in efficiency units.

²² The model does not include equation (T-3) as it has been substituted throughout to minimize the creation of additional variables.

$$(T-1) \quad XAT_{r,i} = \sum_{aa} \gamma_{r,i,aa}^c XA_{r,i,a}$$

$$(T-2) \quad PA_{r,i,aa} = (1 + \tau_{r,i,aa}^{Ap}) \gamma_{r,i,aa}^c PAT_{r,i}$$

$$(T-3) \quad PA_{r,i,aa}^a = PA_{r,i,aa} + \sum_{em} \tau_{r,em}^{emi} \phi_{r,em,i,aa} \rho_{r,em,i,aa}$$

As described above, the decomposition of the Armington aggregate, XAT , is done at the national level. (The Armington equations are all indexed by im . The model allows for homogeneous traded commodities and these are indexed by ih .) Aggregate national demand for domestic goods, XD , is then a fraction of XAT , with the fraction sensitive to the relative price of domestic goods, PD , to the Armington good, PAT —as shown in equation (T-4). The key parameter, known as the Armington substitution elasticity, is σ^m . The model allows for quality differences in the Armington composite goods using the γ^a and γ^t parameters. These in effect allow one to calibrate the CES functions in terms of value shares with the appropriate initialization of the respective γ parameters. Equation (T-5) determines the demand for aggregate imports, XMT , which are further decomposed by trading partner (see below). The price of aggregate imports is tariff-inclusive. Finally, equation (T-6) defines the aggregate (or national) price of the aggregate Armington good, PAT .

$$(T-4) \quad XD_{r,im} = \alpha_{r,im}^d (\gamma_{r,im}^a)^{\sigma_{r,im}^m - 1} \left(\frac{PAT_{r,im}}{PD_{r,im}} \right)^{\sigma_{r,im}^m} XAT_{r,i,a}$$

$$(T-5) \quad XMT_{r,im} = \alpha_{r,im}^m (\gamma_{r,im}^t)^{\sigma_{r,im}^m - 1} \left(\frac{PAT_{r,im}}{PMT_{r,im}} \right)^{\sigma_{r,im}^m} XAT_{r,i,a}$$

$$(T-6) \quad PAT_{r,im} = \left[\alpha_{r,im}^d \left(\frac{PD_{r,im}}{\gamma_{r,im}^a} \right)^{1 - \sigma_{r,im}^m} + \alpha_{r,im}^m \left(\frac{PMT_{r,im}}{\gamma_{r,im}^t} \right)^{1 - \sigma_{r,im}^m} \right]^{1/(1 - \sigma_{r,im}^m)}$$

Each bilateral trade flow is associated with four different prices:

1. PE represents the factory or farm gate price
2. WPE represents the FOB price, an export tax or subsidy induces a wedge between the producer price and the FOB price²³
3. WPM represents the CIF price, international trade and transport margins introduce a wedge between the FOB and CIF price
4. PM represents the agent-price and includes the bilateral tariff

²³ The ENVISAGE model specification of export taxes is that they are an ad valorem tax on the producer price, thus an export subsidy is negative. An alternative formulation would be to specify the tax as a wedge between the world price and the domestic FOB price in which case the subsidy is measured as a positive wedge.

Equations (T-7) through (T-9) describe three of the prices associated with international trade, respectively WPE , WPM and PM (the determination of PE is described below). The respective wedges are represented by τ^e , the export tax/subsidy, τ^{im} , the international transport margin, and τ^m the bilateral tariff. The price of a unit of international transport is uniform, irrespective of the transport node and sector.

Second level Armington nest

The second nest in the Armington structure allocates aggregate import demand (across all agents) to specific regions of origin. The bilateral trade flow will reflect preferences, the region of origin-specific export price and the bilateral tariff, τ^m . The price impacts are reflected in the tariff-inclusive bilateral price PM . Equation (T-10) defines import demand, WTF^d , by region r , sourced in region r' . Equation (T-11) defines the aggregate import price, PMT . It is an aggregation of the tariff inclusive bilateral import price. All agents are assumed to face the same import price (net of the sales tax), i.e. implicitly we are assuming that the composition of the import bundle by each agent is identical.

$$(T-7) \quad WPE_{r,r',im} = (1 + \tau_{r,r',im}^e) PE_{r,r',im}$$

$$(T-8) \quad WPM_{r,r',im} = WPE_{r,r',im} + \tau_{r,r',im}^{im} PWMG$$

$$(T-9) \quad PM_{r,r',im} = (1 + \tau_{r,r',im}^m) WPM_{r,r',im}$$

$$(T-10) \quad WTF_{r',r,im}^d = \alpha_{r',r,im}^w (\gamma_{r',r,im}^m)^{\sigma_{r,im}^w - 1} \left(\frac{PMT_{r,im}}{PM_{r',r,im}} \right)^{\sigma_{r,im}^w} XMT_{r,im}$$

$$(T-11) \quad PMT_{r,im} = \left[\sum_{r'} \alpha_{r',r,im}^w \left(\frac{PM_{r',r,im}}{\gamma_{r',r,im}^m} \right)^{1 - \sigma_{r,im}^w} \right]^{1/(1 - \sigma_{r,im}^w)}$$

Export supply

The ENVISAGE model allows for imperfect transformation of output across markets of destination—domestic and for export. A two-nested CET structure is implemented. At the top level, output is allocated between the domestic market and aggregate exports. At the next level, aggregate exports are allocated across various foreign markets. At either nest, infinite transformation is allowed in which case the CET first order conditions are replaced by the law of one price. The supply of international trade and transport services (XMG) is treated apart and is assumed to be priced at the average producer price, PP .

Equations (T-12) and (T-13) represent the derived supply for domestic, XDT , and aggregate export, XET , markets respectively. With finite transformation, these conditions are the standard

CET first order conditions based on supply (less supply of international trade and transport services). With perfect transformation, each is replaced with the law of one price whereby the domestic, PD , and export, PET , producer price are set equal to the aggregate supply price, PS . Equation (T-14) represents the market equilibrium for supply. With perfect transformation domestic supply is equal to the sum of supply to the various markets—domestic, XDT , aggregate exports, XET , and international trade and transport services, XMG . With finite transformation, the aggregation function is equal to the CET primal function. However, this can be replaced with the CET dual price function as is the case in equation (T-14). All equations allow for a component specific quality or efficiency factor, γ^d and γ^e .

$$(T-12) \quad \begin{cases} PD_{r,im} = \gamma_{r,im}^d PS_{r,im} & \text{if } \sigma_{r,im}^x = \infty \\ XDT_{r,im}^s = \gamma_{r,im}^{xd} (\gamma_{r,im}^d)^{\sigma_{r,im}^x - 1} \left(\frac{PD_{r,im}}{PS_{r,im}} \right)^{\sigma_{r,im}^x} [XS_{r,im} - XMG_{r,im}] & \text{if } \sigma_{r,im}^x < \infty \end{cases}$$

$$(T-13) \quad \begin{cases} PET_{r,im} = \gamma_{r,im}^e PS_{r,im} & \text{if } \sigma_{r,im}^x = \infty \\ XET_{r,im} = \gamma_{r,im}^{xe} (\gamma_{r,im}^e)^{\sigma_{r,im}^x - 1} \left(\frac{PET_{r,im}}{PS_{r,im}} \right)^{\sigma_{r,im}^x} [XS_{r,im} - XMG_{r,im}] & \text{if } \sigma_{r,im}^x < \infty \end{cases}$$

$$(T-14) \quad \begin{cases} XS_{r,im} = XDT_{r,im}^s + XET_{r,im} + XMG_{r,im} & \text{if } \sigma_{r,i}^x = \infty \\ PS_{r,i} = \left[\gamma_{r,im}^{xd} \left(\frac{PD_{r,im}}{\gamma_{r,im}^d} \right)^{1+\sigma_{r,im}^x} + \gamma_{r,im}^{xe} \left(\frac{PET_{r,im}}{\gamma_{r,im}^e} \right)^{1+\sigma_{r,im}^x} \right]^{1/(1+\sigma_{r,im}^x)} & \text{if } \sigma_{r,i}^x < \infty \end{cases}$$

Equations (T-15) and (T-16) reflect the second level CET nest allocating aggregate exports, XET , across various export markets as represented by WTF^s . With perfect transformation, the bilateral export producer price is equal to the aggregate export price, PET , and aggregate export supply is simply the sum across all export markets. With finite transformation, the CET first-order condition determines WTF^s and the aggregate export price is the CET aggregation of the regional export prices. Similar to the other trade equations, a quality or efficiency parameter is introduced that allows for prices to deviate from uniformity even with an infinite transformation elasticity.

$$(T-15) \quad \begin{cases} PE_{r,r',im} = \gamma_{r,r',im}^w PET_{r,im} & \text{if } \sigma_{r,im}^z = \infty \\ WTF_{r,r',im}^s = \gamma_{r,r',im}^{xw} (\gamma_{r,r',im}^w)^{\sigma_{r,im}^z - 1} \left(\frac{PE_{r,r',im}}{PET_{r,im}} \right)^{\sigma_{r,im}^z} XET_{r,im} & \text{if } \sigma_{r,im}^z < \infty \end{cases}$$

$$(T-16) \quad \begin{cases} XET_{r,im} = \sum_i \gamma_{r,r',im}^w WTF_{r,r',im}^s & \text{if } \sigma_{r,im}^z = \infty \\ PET_{r,im} = \left[\sum_{r'} \gamma_{r,r',im}^w \left(\frac{PE_{r,r',im}}{\gamma_{r,r',im}^w} \right)^{1+\sigma_{r,im}^z} \right]^{1/(1+\sigma_{r,im}^z)} & \text{if } \sigma_{r,im}^z < \infty \end{cases}$$

Homogeneous traded goods

The model allows for homogeneous traded goods. In principle, none of the goods in GTAP can be treated immediately as homogeneous goods since there exists bilateral trade for all goods. In principle, some goods are nearly homogeneous since either imports or exports are so small that they could be ignored in an intermediate step that moves from the Armington specification to one based on net trade. It is also possible to introduce new commodities into the model as either Armington or homogeneous goods.

Equation (T-17) defines net trade, NT , for homogeneous goods defined over index ih . It is defined as a value and is the difference between domestic supply, XS , and domestic demand, XAT , evaluated at the world price, PW . Net trade is negative if demand exceeds supply. Equation (T-18) is the market equilibrium condition for homogeneous goods. At equilibrium, the sum of net trade across all countries must equal 0. Equation (T-19) determines the domestic price of homogeneous goods—it is equal to the world price adjusted by the tariff (that is no longer region of origin specific). Both supply and demand prices are equal as provided by equation (T-20).

$$(T-17) \quad NT_{r,ih} = PW_{ih} (XS_{r,ih} - XAT_{r,ih})$$

$$(T-18) \quad \sum_r NT_{r,ih} = 0$$

$$(T-19) \quad PS_{r,ih} = (1 + \tau_{r,ih}^m) PW_{ih}$$

$$(T-20) \quad PAT_{r,ih} = PS_{r,ih}$$

The next three equations are not strictly necessary for the model, but provide identities that can be useful. The first, (T-21), defines the volume of aggregate exports. It is specified as a mixed complementarity formula, or using an orthogonality condition. For the relation to hold, exports must be equal to net trade, or if net trade is negative, exports are set to zero, i.e. they must never fall below zero. The second equation (T-22), almost identical, defines the aggregate volume of imports. If exports are positive, XMT will be zero, else, exports are equal to the negative of net

trade and will be positive. The third is a definition of a world price for Armington goods, and is a weighted global sum of bilateral trade with weights evaluated at previous year trade shares (or at base year trade shares in the case of the comparative static version of the model).

$$(T-21) \quad (PW_{ih} XET_{r,ih} - NT_{r,ih}).XET_{r,ih} = 0 \quad \text{and} \quad XET_{r,ih} \geq 0$$

$$(T-22) \quad NT_{r,ih} = PW_{ih} (XET_{r,ih} - XMT_{r,ih})$$

$$(T-23) \quad PW_{im,t} = \frac{\sum_{r'} \sum_{im} WPE_{r,r',im,t} WTF_{r,r',im,t-1}}{\sum_{r'} \sum_{im} WPE_{r,r',im,t-1} WTF_{r,r',im,t-1}}$$

Domestic supply

The model allows for multi-output production activities (for example producing ethanol and DDGS from ethanol production) and the aggregation of goods produced by activities into a single commodity (for example different streams of electrical production—coal, gas, hydro, nuclear, renewables, etc.—each with their own cost structure, but combined by a distributor into a single commodity).

Activity a can therefore produce a suite of commodities indexed by i , hence an output at this level is indexed by both a and i , $X_{a,i}$.²⁴ This is implemented using a CET structure with the possibility of infinite transformation. Equation (T-24) defines the supply of $X_{a,i}$ emanating from activity a (or XP_a), where the law of one price holds in the case of a finite transformation. Equation (T-25) represents the zero profit condition, or the revenue balance for the multi-output production function.

$$(T-24) \quad \begin{cases} X_{r,a,i} = \gamma_{r,a,i}^P \left(\frac{P_{r,a,i}}{PP_{r,a}} \right)^{\omega_{r,a}^s} XP_{r,i} & \text{if } \omega_{r,a}^s < \infty \\ P_{r,a,i} = PP_{r,a} & \text{if } \omega_{r,a}^s = \infty \end{cases}$$

$$(T-25) \quad PP_{r,a} XA_{r,a} = \sum_{i \in \{\gamma_{r,a,i}^P \neq 0\}} P_{r,a,i} X_{r,a,i}$$

In the next step, multiple streams of output can be combined into a single supplied commodity, XS_i , with a CES-aggregator. The specification allows for homogeneous goods, for example electricity—in which case the cost of each component must be equal, subject perhaps to an efficiency differential. Equation (T-26) determines the demand for produced commodity X . In

²⁴ In the GTAP database this will be represented by a diagonal matrix where each activity produces one and only one good.

the case of a finite elasticity it is a CES formulation. With an infinite substitution elasticity, the law-of-one price must hold, i.e. the producer price of each component must be equalized in efficiency units. Equation (T-27) determines the equilibrium condition in the form of the cost function equality.

$$(T-26) \quad \begin{cases} X_{r,a,i} = \alpha_{r,a,i}^s (\gamma_{r,a}^s)^{\sigma_{r,i}^s - 1} \left(\frac{PS_{r,i}}{P_{r,a,i}} \right)^{\sigma_{r,i}^s} XS_{r,i} & \text{if } \sigma_{r,i}^s < \infty \\ P_{r,a,i} = \gamma_{r,a}^s PS_{r,i} & \text{if } \sigma_{r,i}^s = \infty \end{cases}$$

$$(T-25) \quad PS_{r,i} XS_{r,i} = \sum_{a \in \{\alpha_{r,a,i}^s \neq 0\}} P_{r,a,i} X_{r,a,i}$$

International trade and transport services

The global demand for international trade and transport services will be driven by the overall level of trade. Its allocation across suppliers is specified as a CES function where demand (partially) adjusts to the low-cost suppliers. Within each region, production of these services is given by a CES technology.

Equation (T-28) determines the global demand for international trade and transport services, $XWMG$.²⁵ Regional supply of these services, $XTMG$, is determined in equation (T-29), the CES first order conditions. The global price, $PWVG$, is given in equation (T-30), the CES dual price formula. The regional supply price, $PTMG$, is given in equation (T-31). And the sectoral and regional supply of these services, XMG , is given in equation (T-32).

²⁵ Note that the current formulation assumes that homogeneous goods are transported at no cost internationally.

$$(T-28) \quad PWMG.XWMG = \sum_r \sum_{r'} \sum_{im} (WPM_{r,r',im} - WPE_{r,r',im}) WTF_{r,r',im}^s$$

$$(T-29) \quad XTMG_r = \alpha_r^{mg} \left(\frac{PWMG}{PTMG_r} \right)^{\sigma_r'} XWMG$$

$$(T-30) \quad PWMG = \left[\sum_r \alpha_r^{mg} PTMG_r^{1-\sigma_r'} \right]^{1/(1-\sigma_r')}$$

$$(T-31) \quad PTMG_r = \left[\sum_i \alpha_{r,i}^{mg} PP_{r,i}^{1-\sigma_r''} \right]^{1/(1-\sigma_r'')}$$

$$(T-32) \quad XMG_{r,i} = \alpha_{r,i}^{mg} \left(\frac{PTMG_r}{PP_{r,i}} \right)^{\sigma_r''} XTMG_r$$

Product market equilibrium

The model has only two ‘basic’ commodities—domestically produced goods for the domestic market, XDT , and bilateral exports, WTF . All other goods are composite goods. Equations (G-1) and (G-2) determine the equilibrium price for these two sets of goods, respectively PD and PE . With perfect transformation (at both levels), the true goods market equilibrium price is PS and equation (T-14) is the market equilibrium condition. In the model implementation, the equilibrium conditions (G-1) and (G-2) are substituted out.

$$(G-1) \quad XDT_{r,i}^d = XDT_{r,i}^s$$

$$(G-2) \quad WTF_{r,r',i}^d = WTF_{r,r',i}^s$$

Factor market equilibrium

Economy-wide factor markets

In the standard version of ENVISAGE labor and land markets are national, i.e. economy-wide markets ranging from no mobility to full mobility. In the comparative static model, capital markets are treated the same way, but the dynamic version of the model, with vintage capital, has a somewhat different structure. This section first treats with the national markets. The following section describes the market for sector-specific factors. And the final section deals with capital markets in the dynamic model.

Clearance on national markets is governed by the degree of mobility across sectors and is modeled using a constant-elasticity-of-transformation specification. With an infinite transformation elasticity, factors of production are perfectly mobile across sectors and the law of

one price holds. With finite (and even zero) transformation elasticity, factors are only partially mobile (or sector-specific) and factor returns are sector specific.

With perfectly mobile factors, equation (F-1) determines the market-clearing price where (exogenous) aggregate supply, XFT , is equal to the sum of factor demand across all sectors. The index fp covers all nationally allocated factors of production—unskilled and skilled labor, land, and capital in the comparative static version of the model. The law of one price holds, and thus the sectoral (net of tax) return, NPF , is equal to the economy-wide return, PFT . Equation (F-3) trivially sets sectoral supply equal to sectoral demand. In the case of finite transformation, sectoral factor supply, XF^s , is determined by the CET first order condition, equation (F-2), with aggregate supply, XFT , still exogenous. With zero transformation elasticity, the factor becomes sector-specific. Equation (F-1) then determines the aggregate price, PFT , using the CET dual price formula. The sector-specific equilibrium price, NPF , is determined in the market clearing condition given by equation (F-3). Note that in the GAMS implementation, equation (F-3) is substituted out.

[Need to modify this section to allow for segmented labor markets and eventually for differentiated land markets...]

$$(F-1) \quad \begin{cases} XFT_{r,fp} = \sum_a XF_{r,fp,a}^d & \text{if } \omega_{r,fp}^f = \infty \\ PFT_{r,fp} = \left[\sum_a \alpha_{r,fp,a}^{fs} NPF_{r,fp,a}^{1+\omega_{r,fp}^f} \right]^{1/(1+\omega_{r,fp}^f)} & \text{if } \omega_{r,fp}^f < \infty \end{cases}$$

$$(F-2) \quad \begin{cases} NPF_{r,fp,a} = PFT_{r,fp} & \text{if } \omega_{r,fp}^f = \infty \\ XF_{r,fp,a}^s = \alpha_{r,fp,a}^{fs} \left(\frac{NPF_{r,fp,a}}{PFT_{r,fp}} \right)^{\omega_{r,fp}^f} XFT_{r,fp} & \text{if } \omega_{r,fp}^f < \infty \end{cases}$$

$$(F-3) \quad XF_{r,fp,a}^s = XF_{r,fp,a}^d$$

Sector-specific factor markets

The sector specific factor—normally the natural resource base in natural resource sectors—is handled using an upward sloping supply curve with the elasticity given by ε^{ff} .²⁶ If the latter is infinite, the return to the sector specific factor is assumed to rise at the same rate as the GDP deflator, see equation (F-4). The finite supply curve has three shifters. The first, α^{fs} , is calibrated with base year data. The second, α^{rfs} , can be calibrated in a dynamic scenario to target a region specific variable, for example output or the regional producer price. The third, α^{gfs} , can be calibrated in a dynamic scenario to target a global variable, for example global output or the global price. In this case, the shifter moves each country/regional supply curve by the same

²⁶ A future version of the model will include a resource depletion module for natural gas and crude oil.

proportional amount. Equation (F-5) provides the equilibrium condition for the sector specific factor equating supply to demand.

$$(F-4) \quad \begin{cases} XF_{r,a}^s = \alpha_{r,a}^{rfs} \alpha_a^{gfs} \alpha_{r,a}^{fs} \left(\frac{NPF_{r,NatRs,a}}{PGDPMP_r} \right)^{\varepsilon_{r,a}^{ff}} & \text{if } \varepsilon_{r,a}^{ff} < \infty \\ NPF_{r,NatRs,a} = PGDPMP_r \cdot NPF_{r,NatRs,a,-1} & \text{if } \varepsilon_{r,a}^{ff} = \infty \end{cases}$$

$$(F-5) \quad XF_{r,a}^s = XF_{r,NatRs,a}^d$$

Capital markets with the vintage capital specification

This section describes sectoral capital allocation under the assumption of multiple vintage capital. Capital market equilibrium under the vintage capital framework assumes the following:

- *New* capital is perfectly mobile and its allocation across sectors insures a uniform rate of return.
- *Old* capital in expanding sectors is equated to new capital, i.e. the rate of return on *Old* capital in expanding sectors is the same as the economy-wide rate of return on new capital.
- Declining sectors release *Old* capital. The released *Old* capital is added to the stock of *New* capital. The assumption here is that declining sectors will first release the most mobile types of capital, and this capital, being mobile, is comparable to *New* capital (e.g. transportation equipment).
- The rate of return on capital in declining sectors is determined by sector-specific supply and demand conditions.

The result of these assumptions is that if there are no sectors with declining economic activity, there is a single economy-wide rate of return. In the case of declining sectors, there will be an additional sector-specific rate of return on *Old* capital for each sector in decline.

To determine whether a sector is in decline or not, one assesses total sectoral demand (which of course, in equilibrium equals output). Given the capital-output ratio, it is possible to calculate whether the initially installed capital is able to produce the given demand. In a declining sector, the installed capital will exceed the capital necessary to produce existing demand. These sectors will therefore release capital on the secondary capital market in order to match their effective (capital) demand with supply. The supply schedule for released capital is a constant elasticity of supply function where the main argument is the change in the relative return between *Old* and *New* capital. Supply of capital to the declining sector is given by the following formula:

$$K_{a,Old}^s = K_a^0 \left[R_{a,Old} / R_{a,New} \right]^{\eta^k}$$

where $K_{a,Old}^s$ is capital supply in the declining sector, K_a^0 is the initial installed (and depreciated) capital in the sector at the beginning of the period, and η^k is the dis-investment elasticity. (Note that in the model, the variable R is represented by PF .) In other words, as the rate of return on

Old capital increases towards (decreases from) the rate of return on *New* capital, capital supply in the declining sector will increase (decrease). Released capital is the difference between K^0 and $K^{s,Old}$. It is added to the stock of *New* capital. In equilibrium, the *Old* supply of capital must equal the sectoral demand for capital:

$$K_{a,Old}^s = KV_{a,Old}$$

Inserting this into the equation above and defining the following variable

$$RR_a = R_{a,Old} / R_{a,New}$$

yields the following equilibrium condition:

$$KV_{a,Old} = K_a^0 [RR_a]^{\eta_a^k}$$

The supply curve is kinked, i.e. the relative rate of return is bounded above by 1. If demand for capital exceeds installed capital, the sector will demand *New* capital and the rate of return on *Old* capital is equal to the rate of return on *New* capital, i.e. the relative rate of return is 1. The kinked supply curve has been transformed into a mixed complementarity (MCP) relation. The following inequality is inserted in the model:

$$K_{a,Old}^s = K_a^0 [RR_a]^{\eta_a^k} \leq K_a^{d,Not} = \chi_a^v XP_a$$

The right-hand side determines the *notional* demand for capital in sector a , i.e. it assesses aggregate output (equal to demand) and multiplies this by the capital output ratio for *Old* capital. This is then the derived demand for *Old* capital. If the installed capital is insufficient to meet demand for *Old* capital, the sector will demand *New* capital, and the inequality obtains with the relative rates of return capped at 1. If the derived demand for *Old* capital is less than installed capital, the sector will release capital according to the supply schedule. In this case the inequality transforms into an equality, and the relative rate of return is less than 1.

Equation (F-6) determines the capital output ratio, χ^v for *Old* capital. Equation (F-7) specifies the supply schedule of *Old* capital. In effect, this equation determines the variable RR , the relative rate of return between *Old* and *New* capital.

$$(F-6) \quad \chi_{r,a}^v = \frac{KV_{r,a,Old}}{XP_{r,a,Old}}$$

$$(F-7) \quad K_{r,a}^0 (RR_{r,a})^{\eta_a^k} \leq \chi_{r,a}^v XP_{r,a} \quad \text{and} \quad RR_{r,a} \leq 1$$

There is a single economy-wide rate of return on *New* capital. The equilibrium rate of return on *New* capital is determined by setting aggregate supply equal to aggregate demand. Aggregate demand for new capital is given by:

$$\sum_{a \in \text{Expanding}} \sum_v KV_{a,v}$$

where the set *Expanding* includes all sectors in expansion. Since *Old* capital in expanding sectors is equated with *New* capital, the appropriate sum is over all vintages. The aggregate capital stock of *New* capital is equal to the total capital stock, less capital supply in declining sectors:

$$K^s - \sum_{a \in \text{Declining}} K_a^{s, \text{Old}}$$

where the set *Declining* covers only those sectors in decline. However, at equilibrium, capital supply in declining sectors must equal capital demand for *Old* capital, and capital demand for *New* capital in these sectors is equal to zero. Hence, the supply of *Old* capital in declining sectors can be shifted to the demand side of the equilibrium condition for *New* capital, and this simplification yields equation (F-9) which determines the economy-wide rate of return on *New* capital. Equation (F-8) adds up capital demand across vintages. Equation (F-10) determines the vintage and sector specific rates of return.²⁷ For *New* capital, *RR* is 1 and thus the rate of return on *New* capital is always equal to the economy-wide rate of return (adjusted by the factor tax). For *Old* capital, if the sector is in decline, *RR* is less than 1 and the rate of return on *Old* capital will be less than the economy-wide rate of return (adjusted by the factor tax).

$$(F-8) \quad XF_{r, \text{Capit}, a} = \sum_v KV_{r, a, v}$$

$$(F-9) \quad \sum_a XF_{r, \text{Capit}, a} = XFT_{r, \text{Capit}}$$

$$(F-10) \quad PKV_{r, a, v} = PFT_{r, \text{Capit}} RR_{r, a} (1 + \tau_{r, \text{Capit}, a}^v)$$

Allocation of Output across Vintages

This section describes how output is allocated across vintages. Aggregate sectoral output, *XP*, is equated to aggregate sectoral demand and is derived from *XS*, which itself is derived from a CET aggregation of *XD* and *XET*. Given the beginning of period installed capital, it is possible to assess the level of *potential* output produced using the installed capital. If this level of output is greater than the aggregate output (demand) level, the sector appears to be in decline, installed capital will be released, *Old* output will be equated with aggregate output (demand), and *New* output is zero. Equation (F-11) equates aggregate output, *XP*, to the sum of output across all vintages. Equation (F-12) determines output that can be derived from installed, or *Old* capital, thus equation (F-11) in essence determines output produced with *New* capital by residual. *Old* output is equated to the sectoral supply of *Old* capital, divided by the capital/output ratio. The final two equations are necessary price identities. Equation (F-13) sets the aggregate price of capital—in both declining and expanding sectors it is equal to the rate of return on *Old* capital. Equation (F-14) links the user-price of capital to the after-tax rate of return on capital.

²⁷ These are the net rates of return after tax. Thus the relative rate of return variable, *RR*, is defined in terms of the net rate of return.

$$(F-11) \quad XP_{r,a} = \sum_a XPv_{r,a,v}$$

$$(F-12) \quad XPv_{r,a,Old} = K_{r,a}^0 (RR_{r,a})^{t_i^k} / \chi_{r,a}^v$$

$$(F-13) \quad PF_{r,Capt,a} = PKV_{r,a,Old}$$

$$(F-14) \quad PF_{r,fp,a} = (1 + \tau_{r,fp,a}^v) NPF_{r,fp,a}$$

Macro closure

Equation (M-1) defines the government accounting balance, S_r^g . It is the difference between revenues and expenditures, the latter including some share of stock-building expenditures. Equation (M-2) defines the investment/savings balance, with aggregate gross investment expenditures on the left-side and total available savings on the right, including the depreciation allowance and adjusted for stock-building expenditures. Equation (M-3) defines the depreciation allowance. The model's price anchor, or numéraire, $PNUM$, is defined in equation (M-4). It is defined as the unit value of manufactured exports from the high-income countries, where the set defined by $Numer$ spans the manufactured sectors. One equation needs to be dropped from the model specification and typically one equation from equation (M-2) is dropped. This in fact represents a global Walras' Law that has global investment equal to global savings. In the standard version of the model foreign savings are fixed relative to the numéraire. Alternative closures are provided in Annex 4.

$$(M-1) \quad S_r^g = YG_r - PC_{r,Gov} XC_{r,Gov} - \psi_{r,Gov}^{stb} \sum_i PA_{r,i,stb} XA_{r,i,stb}$$

$$(M-2) \quad PC_{r,Inv} XC_{r,Inv} = \sum_h S_{r,h}^h + S_r^g + PNUM \cdot S_r^f + DeprY_r - \psi_{r,Inv}^{stb} \sum_i PA_{r,i,stb} XA_{r,i,stb}$$

$$(M-3) \quad DeprY_r = \delta_r PC_{r,Inv} K_{r,t}$$

$$(M-4) \quad PNUM = \frac{\sum_{r \in HIC} \sum_{r'} \sum_{i \in Numer} WPE_{r,r',i} WTF_{r,r',i,0}^s}{\sum_{r \in HIC} \sum_{r'} \sum_{i \in Numer} WPE_{r,r',i,0} WTF_{r,r',i,0}^s}$$

The following block of equations provides the main macroeconomic identities. Equations (M-5) and (M-6) represent nominal and real GDP at market price, $GDPMP$ and $RGDPMP$ respectively. The GDP at market price deflator, $PGDPMP$, is defined in equation (M-7). Per capital real output, $RGDPPC$, is defined in equation (M-8). And the GDP absorption shares, $GDPShr$, are provided in equation (M-9).

$$(M-5) \quad GDPMP_r = \sum_i \left[\sum_{in} PA_{r,i,in} XA_{r,i,in} + PA_{r,i,spb} XA_{r,i,spb} \right] + \sum_{ih} NT_{r,ih} \\ + \sum_{r'} \sum_i \left[WPE_{r,r',im} WTF_{r,r',im} - WPM_{r',r,im} WTF_{r',r,im} \right] + PTMG_r \cdot XTMG_r$$

$$(M-6) \quad RGDPMP_r = \sum_i \left[\sum_{in} PA_{r,i,in,0} XA_{r,i,in} + PA_{r,i,spb,0} XA_{r,i,spb} \right] \\ + \sum_{r'} \sum_i \left[WPE_{r,r',im,0} WTF_{r,r',im} - WPM_{r',r,im,0} WTF_{r',r,im} \right] \\ + \sum_{ih} PW_{ih,0} (XET_{r,ih} - XMT_{r,ih,0}) + PTMG_{r,0} \cdot XTMG_r$$

$$(M-7) \quad PGDPMP_r = GDPMP_r / RGDPMP_r$$

$$(M-8) \quad RGDPPC_r = RGDPMP_r / Pop_r$$

$$(M-9) \quad GDPShr_{r,in} = \frac{PC_{r,in} XC_{r,in}}{GDPMP_r}$$

The default closure rules of the model are as follows:

- Household savings are endogenous and are either driven by the demographic-influenced savings function or as part of the ELES consumer demand system.²⁸
- Government revenues are endogenous and government expenditures, as a share of nominal GDP, are fixed, thus total expenditures are endogenous. The government balance is fixed, in part to avoid problems of financing sustainability. The government balance is achieved with a uniform shift in the household direct tax schedule. This implies that new revenues, for example generated by a carbon tax, would lower direct taxes paid by households.
- Investment is savings driven. Household and government savings were discussed above. Foreign savings, in the default closure are fixed. Thus investment is largely influenced through household savings.²⁹
- The current account, the mirror entry of the capital account, is exogenous. Ex ante changes to trade, for example a rise in the world price of imported oil, is met through ex post changes in the real exchange rate.

Model dynamics are discussed below.

²⁸ The latter may allow for demographics and other factors to influence the ELES parameters between periods in the dynamic setting where ELES parameters may be re-calibrated.

²⁹ Alternative closures are conceivable, for example targeting investment (as a share of GDP) and allowing the household savings schedule adjust to achieve the target.

Climate module

This section describes the climate module that in the current version is virtually identical to that of Nordhaus' Dice model.³⁰ The module's sequence is as follows. First total emissions are derived. These then lead to atmospheric concentrations—emissions directly add to the atmosphere, but concentrations in the atmosphere also interact with the ocean, creating a dynamic process that would continue even in the absence of emissions. The atmospheric concentration has an impact on radiative forcing, i.e. how much of the sun's energy is reflected back to space. And finally, there is a set of equations that links radiative forcing to temperature—and these relations also contain an interaction with the ocean.

Emissions

The first emissions equation, equation (C-1), determines the level of emissions, EMI , of type em for each unit of consumption of commodity i by agent aa , where aa covers all production activities and final demand accounts. It is simply a fixed coefficient with respect to the demand level.³¹ The aggregate emission by region (or country r), $EMITot$, is defined in equation (C-2) and is the double sum over all agents and inputs, with the possibility of an additional exogenous level of emissions, $EMIOth$. The level of global emissions, $EMIGbl$, is the final summation across all countries and regions, with an additional exogenous component not accounted for in the regional models—see equation (C-3). Note that in equation (C-3), $EMIGbl$ is indexed by z , which includes the different sinks for emissions. The modeled sinks include the atmosphere (atmos), the shallow oceans (upocn) and the deep oceans (dpocn). All emissions CO_2 emissions are emitted to the atmosphere.

$$(C-1) \quad EMI_{r,em,i,aa} = \rho_{r,em,i,aa} XA_{r,i,aa}$$

$$(C-2) \quad EMITot_{r,em} = \sum_{aa} \sum_i EMI_{r,em,i,aa} + EMIOth_{r,em}$$

$$(C-3) \quad EMIGbl_{Atmos,CO_2} = \sum_r EMITot_{r,CO_2} + EMIOthGbl_{CO_2}$$

Concentration, forcing and temperature

As noted above, the model contains three sinks for CO_2 emissions—the atmosphere and the upper and deep oceans. These three sinks are indexed by z . In each period, there is a flow of carbon across the three sinks using a 3 x 3 transition matrix, K . Each column of the transition matrix represents the share of the stock in the sink that flows to a different sink. Thus the diagonal element represents the share of the stock that stays in its own sink. The current values of the concentration transition matrix are provided in a more detailed description of the climate module in Annex 5.

³⁰ See Nordhaus (2007).

³¹ The current version of the model covers CO_2 emissions alone, and only those derived from the consumption of fossil fuels.

$$(C-4) \quad Conc_z = K.Conc_{z,-1} + (12/44).EMIGbl_{z,CO_2,-1}$$

$$(C-5) \quad Forc_{atmos} = fCO_2x. \frac{\log_{10}(Conc_{atmos} / ConcPI)}{\log_{10}(2)} + ForcOth$$

$$(C-6) \quad Temp_{zt} = T.Temp_{zt,-1} + \Theta.Forc_{zt}$$

Equation (C-4) determines the concentration level in each sink. The concentration level is equal to its lagged value, multiplied by the transition matrix. In the absence of new emissions, one can determine the long-term equilibrium by multiplying the matrix K n -times, where n is large enough that the transition matrix converges towards a constant matrix. Carbon emissions are entirely added to atmospheric concentration.³² Note that emissions in the model are in terms of CO_2 , whereas concentrations and other relevant parameters are calibrated to carbon emissions. Thus total CO_2 emissions are multiplied by the factor $(12/44)$ to convert CO_2 to carbon.

Equation (C-5) converts atmospheric concentrations to its impact on radiative forcing. Forcing is a logarithmic function (based 10) of concentration with two key parameters. The first is the pre-industrial concentration level, $ConcPI$. The second is the amount of forcing induced by a doubling of concentration from its pre-industrial level, fCO_2x . The relation allows for an exogenous amount of forcing, that could eventually be negative, as is the current case, due to SO_2 emissions.

Temperature, measured as the increment to temperature in $^{\circ}C$ since 1900, like concentration, has interactions between the atmosphere and the oceans. In this case the ocean is treated as a single sink and the subset zt of z covers only *atmos* and *dpocn*. Equation (C-6) provides the link between temperature in the two sinks with their previous respective temperatures, through a transition matrix T , and the incremental impact from forcing through the matrix Θ .³³ The temperature transition and forcing matrices are further developed in Annex 5.

Emission taxes, caps and trade

There are a number of different potential regimes to limit carbon emissions. The simplest is simply to impose a carbon tax, i.e. set the variable τ^{emi} to some value (measured as \$2004 per unit of emitted CO_2). Emission caps can be set on either a single region/country basis, with a differentiated carbon tax across regions/countries, or on a region-wide basis with a uniform carbon tax. Quota regions are indexed by rq and can be assigned one or more countries. Examples of cap and cap and trade scenarios are provided in Annex 6. Equation (C-7) implements emissions caps for each agglomeration of regions subject to a cap (potentially just single country). The sum of emissions across all regions belong to region rq is capped to $EMICap$ (the shifter is explained below). Equation (C-7) determines the regional emissions tax,

³² The variable $EMIGbl$ is a vector defined overall all sinks, but emissions to the two ocean sinks are always 0.

³³ The variable $Forc$ is a vector defined over both sinks but is only non-zero for the atmosphere.

τ^{emiR} , which will be uniform across all countries/regions belonging to the aggregate region. Equation (C-8) then is an accounting identity that equates the country/region tax, τ^{emi} , to the region-wide emissions tax.

$$(C-7) \quad \sum_{r \in rq} EMITot_{r,em} = \chi_{em}^{Cap} EMICap_{rq,em}$$

$$(C-8) \quad \tau_{r,em}^{emi} = \tau_{rq,em}^{emiR}$$

$$(C-9) \quad QuotaY_{r,em}^{emi} = \tau_{r,em}^{emi} [EMIQuota_{r,em} - EMITot_{r,em}] \quad \text{if Cap and Trade is active}$$

The shifter in equation (C-7) allows for additional targeting, for example a cap on global emissions. Say for example one wants to cap global emissions by 20 percent but only impose a cap on Annex I emissions. There is some potential leakage from the cap on Annex I countries—with non-Annex I countries increasing their emissions—because the world price of fossil fuels may decline and because they increase their production of carbon intensive goods for export to the now less competitive Annex I markets. The cap on Annex I countries can then be thought of as setting the burden shares across Annex I countries and the shifter, χ^{Cap} , in equation (C-7) is then endogenous to meet the overall objective, for example capping global emissions.

Equation (C-9) determines the value of the trade in emissions quota when country/region specific quotas, $EMIQuota$, are allocated. The value of the quota is the difference between the quota and actual emissions, $EMITot$, valued at the emissions tax level. Currently, it is assumed that the quota rents are recycled back to the government.

Model Dynamics

Model dynamics is driven by three factors—similar to most neo-classical growth models. Population and labor force growth rates are exogenous and given essentially by the UN Population Division scenario. The labor force growth rate is equated to the growth rate of the working age population, i.e. the population aged between 15 and 64.³⁴

The second factor is capital accumulation. The aggregate capital stock in any given year, $KStock$, is equated to the previous year capital stock, less depreciation at a rate of δ , plus the previous period's volume of investment, XC_{Inv} , see equation (G-1). The latter is influenced by the national savings rate plus foreign savings and, as well, the unit cost of investment. The aggregate capital stock variable takes two forms. The first, $KStock$, is the aggregate capital stock evaluated at \$2004 prices. The second is the 'normalized' aggregate capital stock, XFT . The normalized capital stock is equal to the tax inclusive base year capital remuneration, i.e. the user cost of capital across sectors. It is normalized because its price is set to 1 in the base year. The ratio of the normalized capital stock to the actual capital stock provides a measure of the gross rate of

³⁴ In future work, these assumptions will be linked to other variables influencing both the decision to work (i.e. the labor force participation rate) and the skill level (via assumptions on education).

return to capital. It is assumed that both measures of the capital stock grow at the same rate and hence equation (G-2) that equalizes the ratio of the two measures.³⁵

$$(G-1) \quad KStock_r = (1 - \delta_r) \cdot KStock_{r,-1} + XC_{r,Inv,-1}$$

$$(G-2) \quad XFT_{r,Capit} = (XFT_{r,Capit,0} / KStock_{r,0}) KStock_r$$

$$(G-3) \quad \lambda_{r,l,a}^f = (1 + \pi_{r,a} + \gamma_r^l) \cdot \lambda_{r,l,a,-1}^f$$

The third factor is productivity. There are a number of productivity factors peppered throughout the model. The key productivity factor is λ^f that corresponds to factor productivity. The following assumptions are made regarding productivity:

- Sectors are segmented into three groups—agriculture, manufacturing and services.
- Productivity in agriculture is exogenous and factor neutral. The λ^n and λ^v parameters are set to grow at some exogenous and uniform rate.
- In the other sectors, productivity is labor augmenting only—and is uniform across both skilled and unskilled labor.
- There is a wedge between productivity in manufacturing and services, represented by the factor π in equation (G-3). It is typically assumed that productivity in manufacturing is greater than in services, i.e. π for manufacturing is positive, and it is zero for services.
- In the calibration, or business-as-usual scenario, the uniform productivity factor, γ^l , is calibrated to achieve some target level of per capita growth, at least for some period, including historical validation from the base year to some current year (say from 2004 to 2008), and including some medium term horizon such as 2015. After 2015, the parameter γ^l can be fixed and per capita growth then is an endogenous variable. In most policy scenarios, the γ^l parameter is fixed.
- Energy efficiency is assumed to improve at some exogenous rate that influences the λ^e parameter.
- International trade and transport margins, τ^{tm} , are assumed to improve at some exogenous rate.

Model implementation

³⁵ It is important to use the actual capital stock in the capital accumulation function since the level of investment must correspond to the actual capital stock, not the normalized level.

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Annex 1: The CES/CET function

The CES Function

Because of the frequent use of the constant elasticity of substitution (CES) function, this annex develops some of the properties of the CES, including some of its special cases. The CES function can be formulated as a cost minimization problem, subject to a technology constraint:

$$\min \sum_i P_i X_i$$

subject to

$$V = \left[\sum_i a_i (\lambda_i X_i)^\rho \right]^{1/\rho}$$

where V is the aggregate volume (of production, for example), X are the individual components (“inputs”) of the production function, P are the corresponding prices, and a and λ are technological parameters. The a parameters are most often called the share parameters. The λ parameters are technology shifters. The parameter ρ is the CES exponent, which is related to the CES elasticity of substitution, which will be defined below.

A bit of algebra produces the following derived demand for the inputs, assuming V and the prices are fixed:

$$(1) \quad X_i = \alpha_i (\lambda_i)^{\sigma-1} \left(\frac{P}{P_i} \right)^\sigma V$$

where we define the following relationships:

$$\rho = \frac{\sigma-1}{\sigma} \Leftrightarrow \sigma = \frac{1}{1-\rho} \quad \text{and} \quad \sigma \geq 0$$

$$\alpha_i = a_i^\sigma \Leftrightarrow a_i = \alpha_i^{1/\sigma}$$

and

$$(2) \quad P = \left[\sum_i \alpha_i \left(\frac{P_i}{\lambda_i} \right)^{1-\sigma} \right]^{1/(1-\sigma)}$$

P is called the CES dual price, it is the aggregate price of the CES components. The parameter σ , is called the substitution elasticity. This term comes from the following relationship, which is easy to derive from Equation (1):

$$\frac{\partial (X_i / X_j)}{\partial (P_i / P_j)} \frac{(P_i / P_j)}{(X_i / X_j)} = -\sigma$$

In other words, the elasticity of substitution between two inputs, with respect to their relative prices, is constant. (Note, we are assuming that the substitution elasticity is a positive number).

For example, if the price of input i increases by 10 per cent with respect to input j , the ratio of input i to input j will decrease by (around) σ times 10 per cent.

We can also derive some key elasticities from these relations. First, is the elasticity of the aggregate price with respect to one of the input prices:

$$(3) \quad \frac{\partial P}{\partial P_i} \cdot \frac{P_i}{P} = s_i = \frac{P_i \cdot X_i}{P \cdot X}$$

In other words, the percent change in the aggregate price is equal to the percent change in the component price multiplied by the value share of that component represented by s_i .

The price elasticities, holding volume constants are given by the following formula:

$$(4) \quad \varepsilon_{ij} = \frac{\partial X_i}{\partial P_j} \frac{P_j}{X_i} = \sigma s_j - \sigma \delta_{ij} = \sigma (s_j - \delta_{ij})$$

This implies that all components are gross substitutes.

The Leontief and Cobb-Douglas functions are special cases of the CES function. In the case of the Leontief function, the substitution elasticity is zero, in other words, there is no substitution between inputs, no matter what the input prices are. Equations (1) and (2) become:

$$(1') \quad X_i = \frac{\alpha_i V}{\lambda_i}$$

$$(2') \quad P = \sum_i \alpha_i \frac{P_i}{\lambda_i}$$

The aggregate price is the weighted sum of the input (efficient) prices. The Cobb-Douglas function is for the special case when σ is equal to one. It should be clear from Equation (2) that this case needs special handling. The following equations provide the relevant equations for the Cobb-Douglas:

$$(1'') \quad X_i = \alpha_i \frac{P}{P_i} V$$

$$(2'') \quad P = A^{-1} \prod_i \left(\frac{P_i}{\alpha_i \lambda_i} \right)^{\alpha_i}$$

where the production function is given by:

$$V = A \prod_i (\lambda_i X_i)^{\alpha_i}$$

and

$$\sum_i \alpha_i = 1$$

Note that in Equation (1") the value share is constant, and does not depend directly on technology change.

Calibration

Typically, the base data set along with a given substitution elasticity are used to calibrate the CES share parameters. Equation (1) can be inverted to yield:

$$\alpha_i = \left(\frac{P_i}{P} \right)^\sigma \frac{X_i}{V}$$

assuming the technology shifters have unit value in the base year. Moreover, the base year prices are often normalized to 1, simplifying the above expression to a true value share. Let's take the Armington assumption for example. Assume aggregate imports are 20, domestic demand for domestic production is 80, and prices are normalized to 1. The Armington aggregate volume is 100, and the respective share parameters are 0.2 and 0.8. (Note that the model always uses the share parameters represented by α , not the share parameters represented by a . This saves on compute time since the a parameters never appear explicitly in any equation, whereas α is raised to the power of the substitution elasticity, i.e. α , occurs frequently.)

The CET Function

With less detail, the following describes the relevant formulas for the CET function, which is similar to the CES specification.

$$\max \sum_i P_i X_i$$

subject to

$$V = \left[\sum_i g_i X_i^\nu \right]^{1/\nu}$$

where V is the aggregate volume (e.g. aggregate supply), X are the relevant components (sector-specific supply), P are the corresponding prices, g are the CET (primal) share parameters, and ν is the CET exponent. The CET exponent is related to the CET transformation elasticity, ω via the following relation:

$$\nu = \frac{\omega + 1}{\omega} \Leftrightarrow \omega = \frac{1}{\nu - 1}$$

Solution of this maximization problem leads to the following first order conditions:

$$X_i = \gamma_i \left(\frac{P_i}{P} \right)^\omega V$$

$$P = \left[\sum_i \gamma_i P_i^{1+\omega} \right]^{1/(1+\omega)}$$

where the γ parameters are related to the primal share parameters, g , by the following formula:

$$\gamma_i = g_i^{-\omega} \Leftrightarrow g_i = \left(\frac{1}{\gamma_i}\right)^{1/\omega}$$

Annex 2: The demand systems

[Still being implemented]

The model contains four different possible demand systems for determining household demand for goods and services:

- CDE or constant differences in elasticities—largely derived from the GTAP model
- ELES or extended linear expenditure system
- LES or linear expenditure system
- AIDADS of an implicitly directly additive demand system, an extension of the LES that allows for more plausible Engel behavior

Three of them (CDE, LES, AIDADS) use a two-tiered structure to first allocate income between savings and expenditures on goods and services. The ELES integrates the savings allocation within its specification. All four systems determine the demand for consumer goods that are different from produced goods. A transition matrix approach is subsequently used to convert consumer goods into produced goods.³⁶

The CDE demand system

The *Constant Difference of Elasticities* (CDE) function is a generalization of the CES function, but it allows for more flexibility in terms of substitution effects across goods.³⁷ The starting point is an implicitly additive indirect utility function (see Hanoch 1975) from which we can derive demand using Roy's identity (and the implicit function theorem).

General Form

A dual approach is used to determine the properties of the CDE function. The indirect utility function is defined implicitly via the following expression:

$$(1) \quad V(p, u, Y) = \sum_{i=1}^n \alpha_i u^{e_i} \left(\frac{p_i}{Y} \right)^{b_i} \equiv 1$$

where p is the vector of commodity prices, u is utility, and Y is income. Using Roy's identity and the implicit function theorem³⁸ we can derive demand, x , where v is the indirect utility function (defined implicitly):

³⁶ At the moment, for lack of additional data, the transition matrix is largely diagonal, with the exception of energy goods.

³⁷ More detailed descriptions of the CDE can be found in Hertel et al (1991), Surry (1993) and Hertel (1997).

³⁸ See Varian 1992, p. 109.

$$(2) \quad x_i = -\frac{\partial v}{\partial p_i} / \frac{\partial v}{\partial Y} = -\left(\frac{\partial \mathcal{V}}{\partial p_i} / \frac{\partial \mathcal{V}}{\partial u}\right) / \left(\frac{\partial \mathcal{V}}{\partial Y} / \frac{\partial \mathcal{V}}{\partial u}\right) = -\left(\frac{\partial \mathcal{V}}{\partial p_i} / \frac{\partial \mathcal{V}}{\partial Y}\right)$$

This then leads to the following demand function—that is implemented as equation (25) in the model implementation.

$$(3) \quad x_i = \frac{\alpha_i b_i u^{e_i b_i} \left(\frac{p_i}{Y}\right)^{b_i-1}}{\sum_j \alpha_j b_j u^{e_j b_j} \left(\frac{p_j}{Y}\right)^{b_j}}$$

Elasticities

In order to calibrate the CDE system, it is necessary to derive the demand and income elasticities of the CDE. The algebra is tedious, but not difficult.

The own-price elasticity is given by the following:

$$(4) \quad \varepsilon_i = \frac{\partial x_i}{\partial p_i} \frac{p_i}{x_i} = -\frac{e_i b_i s_i}{\sum_j s_j e_j} + (b_i - 1) + \frac{s_i \sum_j s_j e_j b_j}{\sum_j s_j e_j} - s_i b_i$$

In deriving the elasticity, we make use of the following formula that defines the elasticity of utility with respect to price (and again makes use of the implicit function theorem):

$$(5) \quad \frac{\partial u}{\partial p_i} \frac{p_i}{u} = -\frac{p_i}{u} \left(\frac{\partial \mathcal{V}}{\partial p_i}\right) / \left(\frac{\partial \mathcal{V}}{\partial u}\right) = -\frac{s_i}{\sum_j s_j e_j}$$

The price elasticity of utility is approximately the value share of the respective demand component as long as the weighted sum of the expansion parameters, e , is close to unity. The value share is defined in the next equation:

$$(6) \quad s_i = \frac{p_i x_i}{Y}$$

Letting $\sigma_i = 1 - b_i$ (or $b_i = 1 - \sigma_i$), we can also write:

$$(7) \quad \varepsilon_i = s_i \left[\sigma_i - \frac{e_i (1 - \sigma_i)}{\sum_j s_j e_j} - \frac{\sum_j s_j e_j \sigma_j}{\sum_j s_j e_j} \right] - \sigma_i$$

With σ uniform, we also have:

$$(8) \quad \varepsilon_i = -\frac{s_i e_i (1 - \sigma)}{\sum_j s_j e_j} - \sigma$$

With both e and σ uniform, the formula simplifies to:

$$(9) \quad \varepsilon_i = -s_i(1-\sigma) - \sigma = \sigma(s_i - 1) - s_i$$

Equation (9) reflects the own-price elasticity for the standard CES utility function. Finally, with e uniform but not σ , we have:

$$(10) \quad \varepsilon_i = s_i \left[2\sigma_i - 1 - \sum_j s_j \sigma_j \right] - \sigma_i$$

The derivation of the cross elasticities is almost identical and will not be carried out here. Combining both the own-and cross price elasticities, the matrix of substitution elasticities takes the following form where we use the Kronecker product, δ :³⁹

$$(11) \quad \varepsilon_{ij} = s_j \left[-b_j - \frac{e_i b_i}{\sum_k s_k e_k} + \frac{\sum_k s_k e_k b_k}{\sum_k s_k e_k} \right] + \delta_{ij} (b_i - 1)$$

Again, we replace b by $1-\sigma$, to get:

$$(12) \quad \varepsilon_{ij} = s_j \left[\sigma_j - \frac{e_i(1-\sigma_i)}{\sum_k s_k e_k} - \frac{\sum_k s_k e_k \sigma_k}{\sum_k s_k e_k} \right] - \delta_{ij} \sigma_i$$

For uniform σ , equation (22) takes the form:

$$(13) \quad \varepsilon_{ij} = -\frac{e_i s_j (1-\sigma)}{\sum_k s_k e_k} - \delta_{ij} \sigma$$

And with a uniform s and e , we have:

$$(14) \quad \varepsilon_{ij} = -s_j(1-\sigma) - \delta_{ij} \sigma = \sigma(s_j - \delta_{ij}) - s_j$$

Finally, for a uniform e only, the matrix of elasticities is:

$$(15) \quad \varepsilon_{ij} = s_j \left[\sigma_j - (1-\sigma_i) - \sum_k s_k \sigma_k \right] - \delta_{ij} \sigma_i$$

The income elasticities are derived in a similar fashion:

$$(16) \quad \eta_i = \frac{\partial x_i}{\partial Y} \frac{Y}{x_i} = \frac{1}{\sum_k s_k e_k} \left[e_i b_i - \sum_k s_k e_k b_k \right] - (b_i - 1) + \sum_k b_k s_k$$

For this, we need the elasticity of utility with respect to income:

$$(17) \quad \frac{\partial u}{\partial Y} \frac{Y}{u} = -\frac{Y}{u} \left(\frac{\partial V}{\partial Y} \right) \left(\frac{\partial V}{\partial u} \right) = \frac{1}{\sum_k s_k e_k}$$

Note that for a uniform and unitary e , the income elasticity of utility is 1.

³⁹ δ takes the value of 1 along the diagonal (i.e. when $i=j$) and the value 0 off-diagonal (i.e. when $i \neq j$).

Replacing b with $1-\sigma$, equation (16) can be re-written to be:

$$(18) \quad \eta_i = \frac{1}{\sum_k s_k e_k} \left[e_i(1-\sigma_i) + \sum_k s_k e_k \sigma_k \right] + \sigma_i - \sum_k s_k \sigma_k$$

With a uniform σ , the income elasticity becomes:

$$(19) \quad \eta_i = \frac{1}{\sum_k s_k e_k} \left[e_i(1-\sigma) + \sigma \sum_k s_k e_k \right] = \frac{e_i(1-\sigma)}{\sum_k s_k e_k} + \sigma$$

With e uniform, the income elasticity is unitary, irrespective of the values of the σ parameters.

From the Slutsky equation, we can calculate the compensated demand elasticities:

$$(20) \quad \xi_{ij} = \varepsilon_{ij} + s_j \eta_i = -\delta_{ij} \sigma_i + s_j \left[\sigma_j + \sigma_i - \sum_k s_k \sigma_k \right]$$

The cross-Allen partial elasticity is equal to the compensated demand elasticity divided by the share:

$$(21) \quad \sigma_{ij}^a = \sigma_j + \sigma_i - \sum_k s_k \sigma_k - \delta_{ij} \sigma_i / s_j$$

It can be readily seen that the difference of the partial elasticities is constant, hence the name of *constant difference in elasticities*.

$$(22) \quad \sigma_{ij}^a - \sigma_{il}^a = \sigma_j - \sigma_l$$

With a uniform σ , we revert back to the standard CES where there is equivalence between the CES substitution elasticity and the cross-Allen partial elasticity:

$$(23) \quad \sigma_{ij}^a = \sigma$$

Calibration

Calibration assumes that we know the value shares, the own uncompensated demand elasticities and the income elasticities. The weighted sum of the income elasticities must equal 1, so the first step in the calibration procedure is to make sure Engel's law holds. One alternative is to fix some (or none) of the income elasticities and re-scale the others using least squares. The problem is to minimize the following objective function:

$$\sum_{i \in \Omega} (\eta_i - \eta_i^0)^2$$

subject to

$$\sum_{i \in \Omega} s_i \eta_i = 1 - \sum_{i \notin \Omega} s_i \eta_i$$

where the set Ω contains all sectors where the income elasticity is not fixed, i.e. its complement contains those sectors with fixed income elasticities. The solution is:

$$\eta_i = \eta_i^0 + s_i \frac{1 - \sum_{i \notin \Omega} s_i \eta_i - \sum_{i \in \Omega} s_i \eta_i^0}{\sum_{i \in \Omega} s_i^2} \quad \forall i \in \Omega$$

Calibration of the σ parameters is straightforward given the own elasticities and the input value shares. The first step is to calculate the Allen partial elasticities, these are simply the own elasticities divided by the budget shares:

$$(24) \quad \sigma_{ii}^a = \frac{\varepsilon_{ii}}{s_i}$$

Next, equation (21) is setup in matrix form:

$$(25) \quad \sigma_{ii}^a = A \sigma_i$$

where the matrix A has the form:

$$(26) \quad A = \begin{bmatrix} 2 - \frac{1}{s_1} - s_1 & -s_2 & \dots & -s_n \\ -s_1 & 2 - \frac{1}{s_2} - s_2 & \dots & -s_n \\ \vdots & \vdots & \ddots & \vdots \\ -s_1 & -s_2 & \dots & 2 - \frac{1}{s_n} - s_n \end{bmatrix}$$

or each element of A has the following formula:

$$a_{ij} = \delta_{ij} (2 - 1/s_i) - s_j$$

We can then solve for σ :

$$(27) \quad \sigma_i = A^{-1} \sigma_{ii}^a$$

There is nothing which guarantees the consistency of the calibrated σ parameters, which are meant to be positive. The calculation of the σ parameters depends only on the budget shares and the own-price uncompensated elasticities. If the calibrated σ parameters are not all positive, one could modify the elasticities until consistency is achieved. In practice, problems have occurred when a sector's budget share dominates total expenditure.

The e parameters are derived from Equation (19) and normalizing them so that their share weighted sum is equal to 1. Equation (19) can then be converted to matrix form and inverted:

$$(28) \quad B = \begin{bmatrix} s_1\sigma_1 + (1-\sigma_1) & s_2\sigma_2 & \dots & s_n\sigma_n \\ s_1\sigma_1 & s_2\sigma_2 + (1-\sigma_2) & \dots & s_n\sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ s_1\sigma_1 & s_2\sigma_2 & \dots & s_n\sigma_n + (1-\sigma_n) \end{bmatrix}$$

or

$$b_{ij} = s_j\sigma_j + \delta_{ij}(1-\sigma_i)$$

$$(29) \quad e_i = B^{-1}C_i = B^{-1}\left(\eta_i - \sigma_i + \sum_k s_k\sigma_k\right)$$

Calibration of the α parameters is based on equations (1) and (3). Start first with equation (3) and write it in terms relative to consumption of good 1, i.e.:

$$(30) \quad \frac{x_i}{x_1} = \frac{\alpha_i b_i u^{e_i b_i} \left(\frac{p_i}{Y}\right)^{b_i-1}}{\alpha_1 b_1 u^{e_1 b_1} \left(\frac{p_1}{Y}\right)^{b_1-1}}$$

This equation can be used to isolate α_i :

$$(31) \quad \alpha_i = \frac{x_i}{x_1} \frac{\alpha_1 b_1 u^{e_1 b_1} \left(\frac{p_1}{Y}\right)^{b_1-1}}{b_i u^{e_i b_i} \left(\frac{p_i}{Y}\right)^{b_i-1}}$$

and then inserted into equation (1):

$$(32) \quad \sum_{i=1}^n \alpha_i u^{e_i b_i} \left(\frac{p_i}{Y}\right)^{b_i} = \alpha_1 u^{e_1 b_1} \frac{b_1}{s_1} \left(\frac{p_1}{Y}\right)^{b_1} \left[\sum_{i=1}^n \frac{s_i}{b_i} \right] \equiv 1$$

The final expression in equation (32) can be used to solve for α_1 since the formula must equal 1 by definition:

$$(33) \quad \alpha_1 = u^{-e_1 b_1} \frac{s_1}{b_1} \left(\frac{Y}{p_1}\right)^{b_1} \left[\sum_{i=1}^n \frac{s_i}{b_i} \right]^{-1}$$

Substituting back into equation (31) we get:

$$(34) \quad \alpha_i = \frac{x_i}{b_i} u^{-e_i b_i} \left(\frac{Y}{p_i} \right)^{b_i - 1} \left[\sum_{j=1}^n \frac{s_j}{b_j} \right]^{-1}$$

The final calibration expression is then the following:

$$(35) \quad \alpha_i = \frac{s_i}{b_i} \left(\frac{Y}{p_i} \right)^{b_i} \frac{u^{-e_i b_i}}{\sum_{j=1}^n \frac{s_j}{b_j}}$$

Utility is undefined in the base data and it is easiest to simply set it to 1.

In conclusion, for calibration we need the budget shares, initial prices, total expenditure, income elasticities and the own-price uncompensated elasticities. From this, we can derive base year consumption volumes, the Allen partial substitution elasticities through equation (24), σ (and therefore b) through equation (27) and the inversion of the A-matrix, e through equation (29) and inversion of the B-matrix, and finally α through equation (35).

The ELES demand system

Many models assume separability in household decision making between saving and current consumption. Lluch and Howe⁴⁰ introduced a relatively straightforward extension of the LES to include the saving decision simultaneously with the allocation of income on goods and services, this has become known as the extended linear expenditure system or the ELES. The ELES is based on consumers maximizing their intertemporal utility between a bundle of current consumption and an expected future consumption bundle represented in the form of savings. The ELES has several attractive features. The utility function of the ELES has the following form:

$$(36) \quad u = \prod_i (x_i - \theta_i)^{\mu_i} \left(\frac{S}{P^s} \right)^{\mu_s}$$

with

$$(37) \quad \sum_i \mu_i + \mu_s = 1$$

where u is utility, x is the vector of consumption goods, S is household saving (in value), P^s is the price of saving, and μ and θ are ELES parameters.

The consumer solves the following problem:

$$(38) \quad \max \prod_i (x_i - \theta_i)^{\mu_i} \left(\frac{S}{P^s} \right)^{\mu_s}$$

subject to

⁴⁰ See Lluch (1973) and Howe (1975).

$$(39) \quad \sum_{i=1}^n p_i x_i + S = Y$$

where p is the vector of consumer prices, and Y is disposable income. The demand functions are:

$$(40) \quad x_i = \theta_i + \frac{\mu_i}{p_i} \left(Y - \sum_{j=1}^n p_j \theta_j \right)$$

$$(41) \quad S = \mu_s \left(Y - \sum_{j=1}^n p_j \theta_j \right) = Y - \sum_{j=1}^n p_j x_j$$

The term in parentheses is sometimes called supernumerary income, i.e. it is the income that remains after subtracting total expenditures on the so-called subsistence (or floor) expenditures as represented by the θ parameter. The parameter μ then represents the marginal budget share out of supernumerary income.

From the demand equation we can derive the income and price elasticities:

$$(42) \quad \eta_i = \frac{\mu_i Y}{p_i x_i} = \frac{\mu_i}{s_i} \quad \eta_s = \frac{\mu_s Y}{S} = \frac{\mu_s}{s}$$

$$(43) \quad \varepsilon_i = \frac{\theta_i (1 - \mu_i)}{x_i} - 1 \quad \varepsilon_s = -1$$

$$(44) \quad \varepsilon_{ij} = -\frac{\mu_i \theta_j p_j}{p_i x_i} = -\frac{\mu_i \theta_j p_j}{s_i Y} \quad \varepsilon_{sj} = -\frac{\mu_s \theta_j p_j}{s Y} = -\frac{\theta_j p_j}{Y^*}$$

where s is the average propensity to save. Note that the matrix of elasticities can be collapsed to a single formula using the Kronecker factor:

$$(45) \quad \varepsilon_{ij} = -\frac{\mu_i}{s_i Y} [\delta_{ij} Y^* + p_j \theta_j]$$

where δ takes the value 1 when i equals j and Y^* is supernumerary income.

Welfare

With the addition of saving, the indirect utility function is given by:

$$(46) \quad v(p, Y) = \prod_i \left(\frac{\mu_i}{p_i} Y^* \right)^{\mu_i} \left(\frac{\mu_s}{P^s} Y^* \right)^{\mu_s}$$

or

$$(47) \quad v(p, Y) = \frac{Y^*}{P} \quad \text{where} \quad P = \prod_i \left(\frac{p_i}{\mu_i} \right)^{\mu_i} \left(\frac{P^s}{\mu_s} \right)^{\mu_s}$$

The expenditure function is derived by minimizing the cost of achieving a given level of utility, u . It is set-up as:

$$\min \sum_{i=1}^n p_i x_i + S$$

subject to

$$\prod_i (x_i - \theta_i)^{\mu_i} \left(\frac{S}{P^s} \right)^{\mu_s} = u$$

The final expression for the expenditure function is:

$$(48) \quad E(p, u) = \sum_{i=1}^n p_i \theta_i + uP$$

where

$$(49) \quad P = \prod_i \left(\frac{p_i}{\mu_i} \right)^{\mu_i} \left(\frac{P^s}{\mu_s} \right)^{\mu_s}$$

Calibration

Calibration of the ELES uses the budget share information from the base SAM, including the household saving share. Typically, calibration uses income elasticities for all of the n commodities represented in the demand system and uses equation (42) to derive the marginal budget shares, μ_i . This procedure leads to a residual income elasticity, which in this case is the income elasticity of saving. The derived savings income elasticity may be implausible, in which case adjustments need to be made to individual income elasticities for the goods, or adjustments can be made on the group of goods, assuming some target for the savings income elasticity.

The first step is therefore to calculate the marginal budget shares using the average budget shares and the initial income elasticity estimates.

$$\mu_i = \frac{\eta_i p_i x_i}{Y} = \eta_i s_i$$

The savings marginal budget share is derived from the consistency requirement that the marginal budget shares sum to 1:

$$\mu_s = 1 - \sum_{i=1}^n \mu_i$$

Assuming this procedure leads to a plausible estimate for the savings income elasticity, the next step is to calibrate the subsistence minima, θ . This can be done by seeing that the demand equations, (40), are linear in the θ parameters. Note that in the case of the ELES the system of equation are of full rank because the μ parameters do not sum to 1 (over the n commodities. They only sum to 1 including the marginal saving share. This may lead to calibration problems if the propensity to save is 0, which may be the case in some SAMs with poor households.) The linear system can be written as:

$$C = I\theta + MY - M \Pi \theta$$

where I is an $n \times n$ identity matrix, M is a diagonal matrix with μ_i / P_i on the diagonal, and Π is a matrix, where each row is identical, each row being the transpose of the price vector. The above system of linear equations can be solved via matrix inversion for the parameter θ :

$$\theta = A^{-1}C^*$$

where

$$A = I - M \Pi$$

$$C^* = C - MY$$

The matrices A and C^* are defined by:

$$A = [a_{ij}] = \begin{cases} 1 - \mu_i & \text{if } i = j \\ -\mu_i \frac{p_j}{p_i} & \text{if } i \neq j \end{cases}$$

$$C^* = [c_i] = x_i - \frac{\mu_i Y}{p_i}$$

The A and C^* matrices are simplified if the price vector is initialized at 1:

$$A = [a_{ij}] = \begin{cases} 1 - \mu_i & \text{if } i = j \\ -\mu_i & \text{if } i \neq j \end{cases}$$

$$C^* = [c_i] = x_i - \mu_i Y$$

In GAMS one could invert the system of equations embodied in equation (40) directly by solving for the endogenous θ while holding all of the other variables and parameters fixed.

The AIDADS demand system

Both the CDE and the ELES suffer from relatively poor Engel behavior. In the case of the CDE, income elasticities stay relatively constant at their initial level irrespective of income growth. The ELES has even worse properties as it relatively quickly converges towards a Cobb-Douglas utility function with unitary income elasticities for all goods, even allowing for a population-adjustment to the subsistence parameters. An alternative demand system, known as AIDADS, has received more attention recently⁴¹ and was initially proposed by Rimmer and Powell.⁴² It is a relatively natural extension to the LES function, the latter being a special case of the AIDADS function. The insight of Rimmer and Powell was to allow the marginal propensity term of the LES to be a function of other variables, rather than be a constant as in the LES. This allows for

⁴¹ Hertel and co-authors have been using AIDADS with the GTAP model. See for example Yu et al. (2002).

⁴² See Rimmer and Powell (1992a), Rimmer and Powell (1992b) and Rimmer and Powell (1996).

more complex demand behavior, as well as providing better validation for observed changes in consumption patterns.

Basic formulation

AIDADS starts with the implicitly additive utility function given by:

$$(50) \quad \sum_i U_i(x_i, u) \equiv 1$$

Assume the following functional form for the utility function:

$$(51) \quad U_i = \mu_i \ln\left(\frac{x_i - \theta_i}{Ae^u}\right)$$

where

$$(52) \quad \mu_i = \frac{\alpha_i + \beta_i G(u)}{1 + G(u)}$$

with the restrictions

$$\sum_i \alpha_i = \sum_i \beta_i = 1$$

$$0 \leq \alpha_i \leq 1$$

$$0 \leq \beta_i \leq 1$$

$$\theta_i < x_i$$

Cost minimization implies the following:

$$\min \sum_i p_i x_i$$

subject to

$$(53) \quad \sum_i \mu_k \ln\left(\frac{x_i - \theta_i}{Ae^u}\right) \equiv 1$$

The first order conditions lead to:

$$(54) \quad \lambda \frac{\partial U_i}{\partial x_i} = p_i = \lambda \frac{\mu_i}{x_i - \theta_i} \Rightarrow \lambda \mu_i = p_i x_i - p_i \theta_i$$

summing over i and using the fact that the μ_i sum to unity implies:

$$(55) \quad \lambda = \sum_i p_i x_i - \sum_i p_i \theta_i = Y - \sum_i p_i \theta_i = Y^*$$

where Y is aggregate expenditure, and Y^* , sometimes referred to as supernumerary income, is residual expenditure after subtracting total expenditure on the so-called subsistence minima, θ .

Re-inserting equation (55) into (54) yields the consumer demand equations:

$$(56) \quad x_i = \theta_i + \frac{\mu_i}{p_i} Y^* = \theta_i + \frac{\mu_i}{p_i} \left[Y - \sum_j p_j \theta_j \right]$$

Equation (56) is virtually identical to the ELES demand equation (40) above. Similar to the linear expenditure system (LES), demand is the sum of two components—a subsistence minimum, θ , and a share of supernumerary income. Unlike the LES, the share parameter, μ , is not constant, but depends on the level of utility itself. AIDADS collapses to the LES if each β parameter is equal to the corresponding α parameter, with the ensuing function of utility, $G(u)$, dropping from equation (52).

Elasticities

This section develops the main expressions for the income and price elasticities. These formulas will be needed to calibrate the initial parameters of the AIDADS function.

Income elasticity

To derive further properties of AIDADS requires specifying a functional form for $G(u)$. Rimmer and Powell (1996) propose the following:

$$(57) \quad G(u) = e^u$$

The first step is to calculate the marginal budget share, ρ , defined as:

$$\rho_i = p_i \frac{\partial x_i}{\partial Y}$$

The following expression can be derived from equation (56):

$$\frac{\partial x_i}{\partial Y} = \frac{Y^*}{p_i} \frac{\partial \mu_i}{\partial Y} + \frac{\mu_i}{p_i} \frac{\partial Y^*}{\partial Y} = \frac{Y^*}{p_i} \frac{\partial \mu_i}{\partial u} \frac{\partial u}{\partial Y} + \frac{\mu_i}{p_i}$$

Thus:

$$(58) \quad \rho_i = \mu_i + Y^* \frac{\partial \mu_i}{\partial u} \frac{\partial u}{\partial Y}$$

Expression (58) can be expanded in two steps—first evaluating the partial derivative of the share variable, μ , with respect to utility, and then the more difficult calculation of the partial derivative of u with respect to income. The share formula is:

$$\mu_i = \frac{\alpha_i + \beta_i e^u}{1 + e^u}$$

Its derivative is:

$$(59) \quad \frac{\partial \mu_i}{\partial u} = \frac{(1 + e^u)(\beta_i e^u) - (\alpha_i + \beta_i e^u)e^u}{(1 + e^u)^2} = \frac{e^u(\beta_i - \alpha_i)}{(1 + e^u)^2}$$

Utility and income are combined in implicit form and thus we will invoke the implicit function theorem to calculate the partial derivative of u with respect to Y . First, insert equation (56) into equation (53):

$$\sum_i \mu_i \ln\left(\frac{x_i - \theta_i}{Ae^u}\right) = \sum_i \mu_i \ln\left(\frac{\mu_i Y^*}{Ae^u p_i}\right) = 1$$

Expanding the latter expression yields:

$$(60) \quad f(u, Y) = \sum_i \mu_i \ln\left(\frac{\mu_i}{p_i}\right) + \ln(Y^*) - \ln(A) - u = 1$$

which provides the implicit relation between Y and u . The implicit function theorem states the following:

$$(61) \quad \frac{\partial u}{\partial Y} = -\frac{\partial f}{\partial Y} \left[\frac{\partial f}{\partial u} \right]^{-1}$$

The partial derivative of f with respect to Y is simply:

$$(62) \quad \frac{\partial f}{\partial Y} = \frac{1}{Y^*}$$

The partial derivative of f with respect to u is:

$$(63) \quad \begin{aligned} \frac{\partial f}{\partial u} &= -1 + \sum_i \left[\frac{\partial \mu_i}{\partial u} \ln\left(\frac{\mu_i}{p_i}\right) + \mu_i \frac{p_i}{\mu_i} \frac{\partial \mu_i}{\partial u} \right] \\ &= -1 + \frac{e^u}{(1+e^u)^2} \sum_i \left[\left(\ln\left(\frac{\mu_i}{p_i}\right) + 1 \right) (\beta_i - \alpha_i) \right] \\ &= \frac{e^u}{(1+e^u)^2} \left[\sum_i (\beta_i - \alpha_i) \ln(x_i - \theta_i) - \frac{(1+e^u)^2}{e^u} \right] \\ &= \frac{e^u}{(1+e^u)^2} \Omega^{-1} \end{aligned}$$

where

$$(64) \quad \Omega = \left[\sum_i (\beta_i - \alpha_i) \ln(x_i - \theta_i) - \frac{(1+e^u)^2}{e^u} \right]^{-1}$$

The second line uses equation (59). In the third line, equation (56) substitutes for the expression in the logarithm, and the adding up constraint allows for the deletion of non-indexed variables. Substituting (62) and (63) into (61) yields:

$$(65) \quad \frac{\partial u}{\partial Y} = -\frac{\Omega (1+e^u)^2}{Y^* e^u}$$

Substituting (59) and (65) into (58) yields:

$$\rho_i = \mu_i - (\beta_i - \alpha_i) \Omega$$

The income elasticities are derived from the following expression:

$$(66) \quad \eta_i = \frac{\partial x_i}{\partial Y} \frac{Y}{x_i} = \frac{\partial x_i}{\partial Y} \frac{Y}{x_i} \frac{p_i}{p_i} = \frac{\rho_i}{s_i}$$

where s_i is the average budget share:

$$(67) \quad s_i = \frac{p_i x_i}{Y} = \frac{p_i \theta_i}{Y} + \mu_i \frac{Y^*}{Y}$$

It can also be written as:

$$(68) \quad s_i = \mu_i + \left(\frac{p_i \theta_i - \mu_i \sum_j p_j \theta_j}{Y} \right)$$

Thus the income elasticity, η , is equal to the ratio of the marginal budget share, ρ , and the average budget share, s .

Price elasticity

The matrix of substitution elasticities is identical to the expression for the ELES and has the form:

$$(69) \quad \sigma_{ij} = [\mu_j - \delta_{ij}] \frac{\mu_i Y^*}{s_i s_j Y}$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

It is clear that the matrix is symmetric. The matrix of substitution elasticities is also equal to:

$$(70) \quad \sigma_{ij} = [\mu_j - \delta_{ij}] \frac{\mu_i Y^*}{s_i s_j Y} = \frac{(x_i - \theta_i)(x_j - \theta_j)}{x_i x_j} \frac{Y}{Y^*} - \frac{\delta_{ij}(x_i - \theta_i)}{s_j x_i}$$

The compensated demand elasticities derive from the following:

$$(72) \quad \xi_{ij} = s_j \sigma_{ij} = [\mu_j - \delta_{ij}] \frac{\mu_i Y^*}{s_i Y}$$

Finally, the matrix of uncompensated demand elasticities is given by:

$$(73) \quad \varepsilon_{ij} = \xi_{ij} - s_j \eta_i = [\mu_j - \delta_{ij}] \frac{\mu_i Y^*}{s_i Y} - s_j \eta_i$$

The uncompensated demand elasticities can also be written as:

$$(73') \quad \varepsilon_{ij} = -\frac{\mu_i}{s_i Y} [p_j \theta_j + \delta_{ij} Y^*] + \frac{s_j}{s_i} (\beta_i - \alpha_i) \Omega$$

The first term on the right-hand side is always negative. The second term differs from the LES expression for the uncompensated demand elasticities.⁴³ We can see from expression (73') that the AIDADS specification allows for both gross complementarity and substitution. As well, it allows for luxury goods, i.e. positive own-price demand elasticities should the second term be positive and greater than the first term.

Implementation

Implementation of AIDADS is somewhat more complicated than the LES since the marginal propensity to consume out of supernumerary income is endogenous, and utility is defined implicitly. The following four equations are needed for model implementation:

$$(74) \quad Y^* = Y - \sum_i p_i \theta_i$$

$$(75) \quad x_i = \theta_i + \frac{\mu_i}{p_i} Y^*$$

$$(76) \quad \mu_i = \frac{\alpha_i + \beta_i e^u}{1 + e^u}$$

$$(77) \quad u = \sum_i \mu_i \ln(x_i - \theta_i) - 1 - \ln(A)$$

Equations (74) and (75) are identical to their LES (ELES) counterparts. Equation (76) determines the level of the marginal propensity to consume out of supernumerary income, μ , which is a constant in the case of the LES (ELES). It requires however the calculation of the utility level, u , which is defined in equation (77).

Calibration

[To be updated] Calibration requires more information than the LES. Where the LES has $2n$ parameters to calibrate (subject to consistency constraints), AIDADS has $3n$ parameters (less the consistency requirements)— α , β and θ . The calibration system includes equations (74)-(77) which have $2+2n$ endogenous variables (Y^* , θ , μ , and A). There are no equations for calibrating the α and β parameters. If we have knowledge of the income elasticities, we can add the following equations:

⁴³ Recall that for the LES, the α and β terms are equal and thus the second term drops.

$$(78) \quad \Psi = \frac{1}{\Omega} = \left[\sum_i (\beta_i - \alpha_i) \ln(x_i - \theta_i) - \frac{(1 + e^u)^2}{e^u} \right]$$

$$(79) \quad \eta_i = \frac{\rho_i}{s_i} = \frac{\mu_i - (\beta_i - \alpha_i)\Omega}{s_i} = \frac{\mu_i}{s_i} - \frac{(\beta_i - \alpha_i)}{s_i} \Psi$$

There are an additional $1+n$ equations, solving for Ψ and α . There is need for an additional n equations. Assuming we have knowledge of at least n price elasticities, for example the own-price elasticities, we can add the following equation:

$$(80) \quad \varepsilon_{ii} = -\frac{\mu_i}{s_i Y} [p_i \theta_i + Y^*] + (\beta_i - \alpha_i) \Omega$$

The α and β parameters are not independent, the following restrictions must hold:

$$(81) \quad \sum_i \alpha_i = 1$$

$$(82) \quad \sum_i \beta_i = 1$$

The system is under-determined, there are $5+4n$ equations and $3+4n$ variables. One solution, is to make the own-price elasticities endogenous. In this case, we are adding n variables, but then the system is over-determined. We can minimize a loss function with respect to the price elasticities:

$$(83) \quad L = \sum_i (\varepsilon_i - \varepsilon_i^0)^2$$

where ε^0 represents an initial guess of the own-price elasticities and the calibration algorithm will calculate the endogenous ε in order to minimize the loss function, subject to constraints (78)-(82) and the model equations (74)-(77). The exogenous parameters in the calibration procedure include $p, x, s, Y, \eta, \varepsilon^0$ and u .

Annex 3: Alternative trade specification

[To be completed]

$$(29) \quad XD_{r,i,a} = \alpha_{r,i,a}^d \left(\frac{PA_{r,i,a}}{(1 + \tau_{r,i,a}^{Ad}) PD_{r,i}} \right)^{\sigma_{r,i,a}^m} XA_{r,i,a}$$

$$(30) \quad XM_{r,i,a} = \alpha_{r,i,a}^m \left(\frac{PA_{r,i,a}}{(1 + \tau_{r,i,a}^{Am}) PMT_{r,i}} \right)^{\sigma_{r,i,a}^m} XA_{r,i,a}$$

$$(31) \quad PA_{r,i,a} = \left[\alpha_{r,i,a}^d \left((1 + \tau_{r,i,a}^{Ad}) PD_{r,i} \right)^{1 - \sigma_{r,i,a}^m} + \alpha_{r,i,a}^m \left((1 + \tau_{r,i,a}^{Am}) PMT_{r,i} \right)^{1 - \sigma_{r,i,a}^m} \right]^{1/(1 - \sigma_{r,i,a}^m)}$$

$$(32) \quad XDT_{r,i}^d = \sum_a XD_{r,i,a}$$

$$(33) \quad XMT_{r,i} = \sum_a XM_{r,i,a}$$

Annex 4: Alternative capital account closures

[To be completed]

...has three different closures for the capital account. The simplest is simply to fix the capital account at base year levels. The second option, as described in HT, is to allow the capital account equilibrate changes in the expected rate of return to capital across regions, i.e. the percentage change of regional rates of return are equal. If returns are equal initially, this is equivalent to assuming perfect international capital mobility. The third option, also described in HT, assumes that the ‘global’ investor has an optimal portfolio initially, and adjusts capital flows to maintain the same portfolio *ex post*.

Equation (58) defines the average rate of return to capital in each region, $AvgRoR$. It is the weighted average of the sectoral rates of return. [? Should the weights be fixed, i.e. indexed by t_0 ?]. The current net rate of return, $RoRC$, is then defined as the average gross regional rate of return, adjusted by changes to the unit cost of capital, and less depreciation—equation (59). Equation (60) defines the motion equation for aggregate capital. The end-of-period capital stock, K_{t+1} , is equal to the beginning period capital stock, K_t , adjusted for depreciation, and augmented by the current period’s volume of investment, XC_{Inv} . The expected rate of return, $RoRE$, is assumed to decline with positive additions to the capital stock. This is the motivation behind equation (61). [See HT for a more detailed description.] Equation (62) defines the value of net investment, $NInv$. Equation (63) defines the average global rate of return, $RoRG$.

The three foreign capital closure rules are encapsulated in equation (64) and are driven by a model flag labeled $KFlowFlag$. The first rule is simply to fix the capital account. To preserve model homogeneity, the initial volume is multiplied by the model numéraire to provide a nominal foreign saving. The second rule equates the percentage change in the expected rate of return in each region. The third rule assumes that global investment is allocated across regions such that the regional composition of capital stocks is invariant. This implies that the percent change in net investment is equal across regions [Shouldn’t we be using as a rule that the capital stock in value terms is proportionately the same across regions]. Equation (64) is defined for all regions except for one. The left out region is indexed by $RSAV$ that is a subset of the set of regions, r . Closure of the model is guaranteed by equation (65) that forces the global sum of the capital flows to be identically equal to zero.

Annex 5—Climate module and further digression on dynamics

Dynamics in a multi-year step

The step size in the model scenarios are allowed to vary across time—in order to save compute time and storage. Particularly in the long-run scenarios, annual increments are not particularly useful. Some of the equations in the model—essentially almost any equation that relies on a lagged variable need to take into account the variable step size, for example equation (G-1), the capital accumulation equation.

$$KStock_r = (1 - \delta_r) \cdot KStock_{r,-1} + XC_{r,inv,-1}$$

In fact, this equation is not even necessary in the model for a step size of 1 since both variables on the right-hand side of the equation are lags. However, let n be the step-size, eventually 1. Then through recursion, the capital accumulation function becomes:

$$KStock_t = (1 - \delta)^n KStock_{t-n} + \sum_{j=1}^n (1 - \delta)^{j-1} XC_{inv,t-j}$$

If the model is run in step sizes greater than 1, the intermediate values of real investment are not calculated. They can be replaced by assuming a linear growth model for investment:

$$XC_{inv,t} = (1 + \gamma^I) XC_{inv,t-1}$$

Replacing this in the accumulation function yields:

$$KStock_t = (1 - \delta)^n KStock_{t-n} + \sum_{j=1}^n (1 - \delta)^{j-1} (1 + \gamma^I)^{n-j} XC_{inv,t-n}$$

With some algebraic manipulation (that is done for a number of similar expressions below), this formula can be reduced to the following:

$$KStock_t = (1 - \delta)^n K_{t-n} + \frac{(1 + \gamma^I)^n - (1 - \delta)^n}{\gamma^I + \delta} XC_{inv,t-n}$$

Where we have the following equation to determine the growth rate of investment:

$$XC_{inv,t} = (1 + \gamma^I)^n XC_{inv,t-n}$$

which itself is now a function of contemporaneous investment. If n is equal to 1, it is clear that this equation simplifies to the simple 1 step accumulation function. The capital accumulation function is no longer exogenous since it depends on the investment growth rate, which itself is endogenous. To avoid scale problems, equations (G-1a) and (G-1b) are used in place of (G-1) to provide the n -step capital stock accumulation function. Equation (G-1a) is likely to evaluate to somewhere between 10 and 20 since the first term is 1 plus the average annual growth of investment, to which is added the depreciation rate less 1. If investment growth is 5% and depreciation is also 5%, then the value is 10. The first term on the right-hand side of equation (G-1b) is likely to be relatively small since it takes the previous capital stock and subtracts a multiple of the previous period's investment (lagged n years), and then multiplies by the

depreciation factor, so that the largest term is the second term, which is a multiple of the current volume of investment.

$$(G-1a) \quad IGFact_{r,t} = \left[\left(\frac{XC_{r,Inv,t}}{XC_{r,Inv,t-n}} \right)^{1/n} - 1 + \delta \right]^{-1}$$

$$(G-1b) \quad KStock_{r,t} = [KStock_{r,t-n} - IGFact_{r,t} XC_{r,Inv,t-n}] (1 - \delta_{r,t})^n + IGFact_{r,t} XC_{r,Inv,t}$$

The savings function, equation (D-8) also needs modification in a dynamic scenario with multiple years between solution periods. The new equation (D-8) below shows the modification of the savings function, where the new variables are g^{PLT15} and g^{P65UP} that represent the average annual growth rates of the youth and elderly dependency ratios.

$$(D-8) \quad \begin{aligned} s_{r,t}^s &= \chi_r^s \alpha_r^s \frac{1 - (\beta_r^s)^{n+1}}{1 - \beta_r^s} + (\beta_r^s)^{n+1} s_{r,t-n}^s + \beta_r^g g_{r,t}^{pc} \frac{1 - (\beta_r^s / (1 + g_r^{pc}))^{n+1}}{1 - \beta_r^s / (1 + g_r^{pc})} \\ &+ \beta_r^y DRAT_{r,t}^{PLT15} \frac{1 - (\beta_r^s / (1 + g_{r,t}^{PLT15}))^{n+1}}{1 - \beta_r^s / (1 + g_{r,t}^{PLT15})} \\ &+ \beta_r^e DRAT_r^{P65UP} \frac{1 - (\beta_r^s / (1 + g_{r,t}^{P65UP}))^{n+1}}{1 - \beta_r^s / (1 + g_{r,t}^{P65UP})} \end{aligned}$$

Dynamics and the climate module

The climate module is directly drawn from Nordhaus' DICE model (see Nordhaus and Boyer 200x and Nordhaus 2007). The transition matrices in the DICE model are based on a fix 10-year time gap between years. In the ENVISAGE model, the time gap is variable. The model therefore requires two modifications to the DICE version of the climate module. First, it is necessary to convert the 10-year transition matrices to a single-year transition matrix, and then to code the dynamic equations (similar to capital accumulation above) to allow for variable gap dynamic expression.

Emissions and concentration

In the DICE model, the 10-year concentration transition matrix B has the following form and values:

$$B = \begin{Bmatrix} & \textit{atmos} & \textit{upocn} & \textit{dpocn} \\ \textit{atmos} & b_{11} & b_{12} & b_{13} \\ \textit{upocn} & b_{21} & b_{22} & b_{23} \\ \textit{dpocn} & b_{31} & b_{32} & b_{33} \end{Bmatrix} = \begin{Bmatrix} & \textit{atmos} & \textit{upocn} & \textit{dpocn} \\ \textit{atmos} & 0.810712 & 0.097213 & 0 \\ \textit{upocn} & 0.189288 & 0.852787 & 0.003119 \\ \textit{dpocn} & 0 & 0.050000 & 0.996881 \end{Bmatrix}$$

Nearly 19% of atmospheric carbon is absorbed by the upper sea (over a decade), and the upper sea releases about 10% of its carbon to the atmosphere (over a decade).

If emissions end at some point T , then the equilibrium concentration of carbon can be given by the following equation:

$$Conc_{\infty} = B^{\infty} Conc_T$$

The equilibrium B matrix, B^{∞} , is given by:

$$B^{\infty} = \begin{pmatrix} & \textit{atmos} & \textit{upocn} & \textit{dpocn} \\ \textit{atmos} & 0.029269 & 0.029269 & 0.029269 \\ \textit{upocn} & 0.056991 & 0.056991 & 0.056991 \\ \textit{dpocn} & 0.913739 & 0.913739 & 0.913739 \end{pmatrix}$$

This implies that in the long run the atmosphere will contain just under 3% of total carbon in all three physical zones (or sinks) as of the terminal year of emissions, with about 6% in the upper ocean and the remaining 91% absorbed in the deep ocean. At today's level of carbon concentrations we would get the following equilibrium concentration levels (assuming all emissions stop today):⁴⁴

$$Conc_{\infty} = \begin{bmatrix} 598 \\ 1,164 \\ 18,667 \end{bmatrix} = B^{\infty} \begin{bmatrix} 809 \\ 1,255 \\ 18,365 \end{bmatrix}$$

This translates into a reduction of 26 percent in atmospheric concentration and a rise of 1.6% in deep ocean concentration. If the entire estimated amount of fossil fuels is spewed out into the atmosphere, over the very long run, the atmospheric concentration would stabilize at 713 GTC, lower than today's level, but in the intermediate years, concentration levels could rise dramatically. In the DICE baseline with no mitigation efforts, concentration levels in the atmosphere max out at around 3,000 GTC in around 2250.

The matrix B is valid for a time horizon spanning 10 years. In other words, equation (C-14) in terms of the DICE model is:

$$Conc_{z,t+10} = BC_{z,t} + E_t$$

where E_t represents the cumulated emissions over 10 years through year t . It is possible to convert B into an annual transition matrix with some matrix algebra and numerical evaluation. If the matrix B is a positive definite matrix, than all of its eigenvalues are positive and it is possible

⁴⁴ The concentration is expressed at the stock of carbon (C) in gigatons.

to take the n^{th} root of the matrix B . The eigenvalues and eigenvectors of a real matrix B solve the following matrix equation:

$$Bx = \lambda x$$

In other words the projection of the vector x , by the matrix B is equal to that same vector multiplied by a scalar, λ . The eigenvalues, λ , can be calculated by solving an n -degree polynomial derived from the determinant of the above system:

$$Bx = \lambda x \Leftrightarrow (B - \lambda I)x = 0 \Rightarrow |B - \lambda I| = 0$$

Let V be the matrix of (right) eigenvectors of B (in columns), and Λ the diagonal matrix composed of the eigenvalues (in the same order as the respective eigenvectors), then B is diagonalized by:

$$B = V\Lambda V^{-1}$$

It can be shown that if Λ has only positive eigenvalues⁴⁵, then the n^{th} root of B can be derived from⁴⁶:

$$B^{1/n} = V\Lambda^{1/n}V^{-1}$$

In the case of the B matrix above, a numerical package has been used to numerically calculate the eigenvalues and eigenvectors⁴⁷:

$$\Lambda = \begin{bmatrix} 0.694258 & 0 & 0 \\ 0 & 0.966122 & 0 \\ 0 & 0 & 1.000000 \end{bmatrix} \quad V = \begin{bmatrix} -0.635745 & -0.311457 & 0.031954 \\ 0.761574 & -0.497912 & 0.062219 \\ -0.125829 & 0.809369 & 0.997551 \end{bmatrix}$$

Thus the annual transition matrix is given by:

⁴⁵ If the diagonal elements of a square matrix B are all positive, and if B and B' are both diagonally dominant, then B is positive definite. The definition of diagonally dominant is that the absolute value of each diagonal element is greater than the sum of absolute values of the non-diagonal elements in its row. That is if for all i $|a(i,i)| > \text{Sum}(|a(i,j)|; j \neq i)$.

⁴⁶ It is pretty easy to see this if $n=2$:

$$C = V\Lambda^{1/2}V^{-1} = B^{1/2} \Leftrightarrow C.C = B \Leftrightarrow V\Lambda^{1/2}V^{-1}.V\Lambda^{1/2}V^{-1} = V\Lambda^{1/2}.\Lambda^{1/2}V^{-1} = B$$

In the next to the last step the square root of the diagonal matrix is simply the square root of each diagonal element and the multiplication of the two diagonal matrices is simply the original diagonal matrix. This is easy to generalize for any integer root.

⁴⁷ The eigenvectors are determined up to a scalar multiple. In the case above, they have been normalized to be on the unit circle.

$$K = B^{1/10} = V\Lambda^{1/10}V^{-1} = \begin{bmatrix} 0.978025 & 0.011566 & -0.000017 \\ 0.022520 & 0.983021 & 0.000338 \\ -0.000545 & 0.005413 & 0.999680 \end{bmatrix}$$

Intuitively, one can see that the diagonal elements of K are roughly equal to the diagonal elements of B raised to the power 0.1 and that the off-diagonal elements are roughly 10% of the off-diagonal elements of B .

[N.B. The third eigenvector reflects the same distribution as the long-run equilibrium distribution described in note 1 above, corresponding to the eigenvalue 1. The equilibrium matrix can also be derived from the following formula:

$$B^\infty = \lim_{n \rightarrow \infty} VK^nV^{-1} = V \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^{-1} = \begin{Bmatrix} 0.029269 & 0.029269 & 0.029269 \\ 0.056991 & 0.056991 & 0.056991 \\ 0.913739 & 0.913739 & 0.913739 \end{Bmatrix}$$

]

Equation (C-4) can then be written in cumulative form as:

$$Conc_t = K^n Conc_{t-n} + \sum_{j=0}^{n-1} K^{n-1-j} E_{t+j-n}$$

Assuming that emissions grow at a compound growth rate of g^e between $t-n$ and t , we have the following:

$$\begin{aligned} Conc_t &= K^n Conc_{t-n} + \sum_{j=0}^{n-1} K^{n-1-j} (1+g)^j E_{t-j-1} \\ &= K^n Conc_{t-n} + (1+g)^{n-1} \sum_{j=0}^{n-1} V\Lambda^{n-1-j} (1+g)^{-n+1+j} V^{-1} E_{t-n} \\ &= K^n Conc_{t-n} + (1+g)^{n-1} V \left[\sum_{j=0}^{n-1} \Lambda^{n-1-j} (1+g)^{-n+1+j} \right] V^{-1} E_{t-n} \end{aligned}$$

The expression within brackets is a diagonal matrix, so it is possible to use standard formulas for a geometric progression to give the following:

$$Conc_t = V\Lambda^n V^{-1} Conc_{t-n} + V\Phi V^{-1} E_{t-n}$$

where Λ is defined as above, and the diagonal matrix Φ is given by:

$$\Phi_{ij} = \begin{cases} \frac{\lambda_i^n - (1 + g^e)^n}{\lambda_i - (1 + g^e)} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Based on these expressions, equation (C-4) that defines concentration for a one period gap is replaced by equations (C-4a), (C-4b) and (C-4c). Equation (C-4a) determines the growth of emissions between period $t-n$ and t assuming a constant annual growth rate, g^{emi} . Equation (C-4b), similar to the expression above, defines a 3 x 3 matrix, *EMIGFact*, which is used to determine the growth factor in the cumulative concentration expression. The parameter λ^c in the expression represents the eigenvalues of the transition matrix K. And equation (C-4c), replacing equation (C-4) determines the cumulative concentration, *Conc*, in period t . The first component on the right is the evolution of the existing stock of carbon concentration where V^c is the matrix of eigenvectors of the K matrix and Λ^c is the diagonal matrix of eigenvalues. The second component represents cumulative emissions over the period n , with adjustments for the transition of the lagged emissions across the sinks.

$$(C-4a) \quad g_{em,t}^{emi} = \left(\frac{EMIGbl_{Armos,em,t}}{EMIGbl_{Armos,em,t-n}} \right)^{1/n} - 1$$

$$(C-4b) \quad EMIGFact_{z,t} = \frac{(\lambda_z^c)^n - (1 + g_{em,t}^{emi})^n}{\lambda_z^c - (1 + g_{em,t}^{emi})}$$

$$(C-4c) \quad Conc_t = V^c (\Lambda^c)^n V^{c-1} Conc_{t-n} + (12/44) \cdot V^c EMIGFact_t V^{c-1} EMIGbl_{z,CO2,t-n}$$

Temperature

The temperature transition, similar to concentration, has to be modified to allow for multiple year gaps. This section describes how the DICE formulation has been adapted for ENVISAGE.

The temperature module in DICE can be collapsed into matrix form:

$$T = M.T_{-1} + B.F = \begin{bmatrix} 1 - \beta_1(\lambda + \beta_2) & \beta_1\beta_2 \\ \beta_3 & 1 - \beta_3 \end{bmatrix} T_{-1} + \begin{bmatrix} \beta_1 \\ 0 \end{bmatrix} F$$

where the transition and impact matrices, M and B are defined for a 10-year transition period. In the steady-state, this can be written as:

$$T^e = [I - M]^{-1} B.F$$

where F is a constant level of radiative forcing. The inverse matrix has a rather simple expression:

$$[I - M]^{-1} = \begin{bmatrix} 1/(\lambda\beta_1) & \beta_2/(\lambda\beta_1) \\ 1/(\lambda\beta_1) & (\lambda + \beta_2)/(\lambda\beta_3) \end{bmatrix}$$

This implies that the equilibrium temperature for both the atmosphere and the deep ocean is given simply by:

$$T^e = F / \lambda$$

With the default value for λ , the equilibrium temperature is about 0.8 times the equilibrium forcing level.

Similar to the concentration equation above, the temperature equation is recursive and can be collapsed into multi-period form by the following formula:

$$T_{t+n} = V\Lambda^n V^{-1}T_t + V\Phi V^{-1}B.F_{t+n}$$

Where Λ is the diagonal matrix of eigenvalues of the one-period transition matrix, and the diagonal matrix Φ is given by:

$$\Phi_{ij} = \begin{cases} \frac{\lambda_i^n - (1 + g^f)^n}{\lambda_i(1 + g^f)^{n-1} - (1 + g^f)^n} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where g^f is the annual compound growth rate of forcing (F) over the n -period range.

There are two differences with the concentration accumulation equation. The first is that the forcing variable, F , is pre-multiplied by the matrix B . The second is that in the DICE code the forcing variable is contemporaneous and not lagged—this changes the accumulation expression compared to the one for concentration. Both nevertheless collapse to 1 when n is equal to 1.

Similar to the concentration matrix, but with additional complications, one must convert the DICE-based 10-period M and B matrices into a 1-period matrix—hopefully preserving as well the particular relations across the different cells of the matrices. The following steps provide one way to do this:

1. Calculate the 10-period eigenvalues and eigenvectors of the 10-period M matrix so that the following holds:

$$M = V\Lambda V^{-1}$$

2. Calculate the 1/10th roots of the eigenvalues and then evaluate the 1-period M matrix, Γ :

$$\Gamma = V\Lambda^{0.1}V^{-1}$$

3. Calculate the β coefficients consistent with the values of the cells in Γ . There are too few degrees of freedom, so some choices must be made. For example:

$$\beta_2 = \frac{\lambda\Gamma_{12}}{1-\Gamma_{11}-\Gamma_{12}} \quad \beta_1 = \frac{\Gamma_{12}}{\beta_2} \quad \beta_3 = \Gamma_{21} \quad \Gamma_{22} = 1 - \beta_3$$

Thus, the bottom right cell of Γ is adjusted so that the sum along the bottom row is 1. The one-period B matrix, B_1 then becomes:

$$B_1 = \beta_1$$

4. Since the Γ matrix has been modified, it is necessary to re-calculate the eigenvalues and eigenvectors consistent with the adjusted 1-period Γ matrix. The new one-period β coefficients and the one-period eigenvalues and eigenvectors can be used for models that use one- or multi-period steps.

Equations (C-5) and (C-6) are then replaced with (C-5) and (C-6a), (C-6b) and (C-6c). In equation (C-5) the only difference is that the expression uses the average concentration between years $t-n$ and t , rather than the concentration of a single year. Equation (C-6a) defines the average annual growth rate in forcing, g^f , between years $t-n$ and t . Equation (C-6b) defines a 2 x 2 diagonal matrix that is used to provide the forcing growth factor, *ForcGFact*, for the cumulative temperature transition equation. It is similar to expression (C-4b) save that the denominator is adjusted to account for the fact that forcing is assumed to impact current temperatures, and not future temperatures, i.e. forcing and temperature are contemporaneous variables. The λ^t parameters are the eigenvalues of the temperature transition matrix. Equation (C-6c) represents the temperature/forcing relation for a multi-year transition period, where V^t is the matrix of eigenvectors of the temperature transition matrix, Λ^t is the matrix of eigenvalues, and Θ is the direct impact of forcing on temperature.

$$(C-5) \quad Forc_{atmos} = fCO2x \cdot \frac{\log_{10}(0.5(Conc_{atmos,t-n} + Conc_{atmos,t}) / ConcPI)}{\log_{10}(2)} + ForcOth$$

$$(C-6a) \quad g_t^f = \left(\frac{Forc_t}{Forc_{t-n}} \right)^n - 1$$

$$(C-6b) \quad ForcGFact_{zt,t} = \frac{(\lambda_{zt}^t)^n - (1 + g_t^f)^n}{\lambda_{zt}^t (1 + g_t^f)^{n-1} - (1 + g_t^f)}$$

$$(C-6c) \quad Temp_{zt,t} = V^t (\Lambda^t)^n V^{t-1} Temp_{zt,t-n} + V^t ForcGFact_t V^{t-1} \Theta Forc_{zt}$$

Annex 6: Examples of emission caps, taxes and tradable permits

[To be completed].

Annex 7: The accounting framework

This annex provides a visual representation of the accounting framework in the GTAP dataset and linked to the model specified in this document. We use the Social Accounting Matrix (or SAM) framework, which is somewhat space consuming, but has the advantage of providing a consistent picture in a single snapshot.⁴⁸ There is no unique representation of a SAM, the one depicted in Figure A7-2 has the advantage of reflecting the accounts in basic prices and with all individual price wedges.

The figure itself does not need much description. There are 18 accounts in total, some of which are purely pass-through accounts (to stick to the tradition that a SAM should be a square and balanced matrix). The following table summarizes the accounts:

Table A7-1: Description of the SAM accounts

<i>Account</i>	<i>Description</i>
ACT	Represents the production activities. The total is domestic output at producer price, the latter includes the producer tax. Revenues are exhausted by payments to intermediate goods (including sales tax) and to factors of production.
COMM	Represents total supply—domestic production and imports. The latter enter at CIF prices, to which are added import taxes. The disposition of total supply includes domestic sales of domestic goods, <i>XD</i> , aggregate imports, exports and supply of international trade and transport services.
DAP	This is the disposition of domestic sales of domestic production at producer price (<i>PD</i>).
MAP	This is the disposition of import sales at tariff inclusive import prices (that are uniform across all agents).
DIT	Revenues generated by the agent-specific sales tax on domestic products.
MIT	Revenues generated by the agent-specific sales tax on imported products.
VA	Value added accounts. In the activity column, it reflects the net of tax cost of the factors of production. All factor remuneration is attributed to the single representative household.
VA_TAX	Revenues from taxes on the factors of production. All tax revenues are attributed to the government account.
PTAX	Output tax revenues.
EXP_TAX	Revenues (or cost) from export taxes (or subsidies).
IMP_TAX	Revenues from import tariffs.
HH	Represents the accounts of the private sector. From a national account perspective, this is a consolidated private sector that includes enterprises and non-governmental organizations. In this SAM, the sole source of income for households is net factor remuneration. Expenditures include demand for goods and services and savings net of depreciation. Households save and pay income taxes to the government. Note that in the SAM database the fiscal accounts are not closed. The model is initialized to assume a zero government deficit and direct taxes represent a residual to balance the household (and government accounts).
GOV	The government collects all indirect taxes and purchases goods and services. Its account is closed by assuming a lump-sum tax on households.
INV	The investment account purchases goods and services. Its income comes from

⁴⁸ For an introduction to Social Accounting Matrices, see Pyatt and Round 1985 and Reinert and Roland-Holst 1997.

domestic private savings gross of depreciation and foreign saving. Public saving is implicitly assumed to be zero.

DEPR
TRADE

The depreciation account is a pass-through account.

The trade account measures the flow of exports (by region of destination) at FOB prices and the flow of imports (by region of origin) at CIF prices. Aggregate exports and imports (across sectors) are recorded in the balance of payment accounts (BoP) by region. The total for these columns/rows is therefore the sum of exports and imports. The difference between exports and imports provides the net trade with each region (though using different prices since exports are evaluated FOB and imports are evaluated COF). Aggregate exports (by region) show up in the BoP column since they represent foreign income. Aggregate imports (by region) show up in the BoP row.

ITT_MARG

This account shows the regional supply of international trade and transport services. Its aggregate sum will show up in the BoP column since it is foreign revenue.

BoP

This account has the consolidated balance of payments. Exports and supply of international trade and transport services will show up as revenues in the column. Imports will show up in the row as a payment to the rest of the world. The balancing item is the capital account that appears in the column as a payment to the investment sector. If it is positive, the region is a net capital importer. If it is negative, the region is a net capital exporter. In the aggregation of all regional SAMs, this item should show up as a zero. Also, the sum of exports across all regions and the sum of international trade and transport services should equal the sum of imports.

Figure A7-2: The schematic Social Accounting Matrix

		ACT	COMM	DAP	MAP	DIT	MIT
		<i>j</i>	<i>j</i>	<i>j</i>	<i>j</i>	<i>j</i>	<i>j</i>
Activities	<i>i</i>		$Diag(PP_{r,j}XP_{r,j})$				
Produced commodities	<i>i</i>			$Diag(PD_{r,j}XDT_{r,j})$	$Diag(PMT_{r,j}XMT_{r,j})$		
Domestic commodities	<i>i</i>	$PD_{r,i}XD_{r,i,j}$					
Imported commodities	<i>i</i>	$PMT_{r,i}XM_{r,i,j}$					
Sales tax on dom. goods	<i>i</i>	$\tau_{r,i,j}^{Ad}PD_{r,i}XD_{r,i,j}$					
Sales tax on imp. goods	<i>i</i>	$\tau_{r,i,j}^{Am}PMT_{r,i}XM_{r,i,j}$					
Value added	<i>fp</i>	$NPF_{r,fp,j}XF_{r,fp,j}$					
Value added tax	<i>fp</i>	$\tau_{r,fp,j}^vNPF_{r,fp,j}XF_{r,fp,j}$					
Production tax	1	$\tau_{r,j}^pPX_{r,j}XP_{r,j}$					
Import tariffs	<i>r'</i>		$\tau_{r',r,j}^mWPM_{r',r,j}WTF_{r',r,j}$				
Export taxes	<i>r'</i>		$\tau_{r,r',j}^ePE_{r,r',j}WTF_{r,r',j}$				
Households	1						
Government	1					$\sum_a \tau_{r,j,a}^{Ad}PD_{r,j}XD_{r,j,a}$	$\sum_a \tau_{r,j,a}^{Am}PMT_{r,j}XM_{r,j,a}$
Investment	1						
Depreciation	1						
Trade	<i>r'</i>		$WPM_{r',r,j}WTF_{r',r,j}$				
International trade margins	1						
Balance of payments	1						

Figure A7-2: The schematic Social Accounting Matrix, continued

		VA	VA_TAX	PTAX	EXP_TAX	IMP_TAX	HH
		fp	fp	1	r'	r'	1
Activities	i						
Produced commodities	i						
Domestic commodities	i						$PD_{r,i}XD_{r,i,h}$
Imported commodities	i						$PMT_{r,i}XM_{r,i,h}$
Sales tax on dom. goods	i						$\tau_{r,i,h}^{Ad}PD_{r,i}XD_{r,i,h}$
Sales tax on imp. goods	i						$\tau_{r,i,h}^{Am}PMT_{r,i}XM_{r,i,h}$
Value added	fp						
Value added tax	fp						
Production tax	1						
Import tariffs	r'						
Export taxes	r'						
Households	1	$\sum_j NPF_{r,fp,j}XF_{r,fp,j}$					
Government	1		$\sum_j \tau_{r,fp,j}^v NPF_{r,fp,j}XF_{r,fp,j}$	$\sum_j \tau_{r,j}^p PX_{r,j}XP_{r,j}$	$\sum_i \tau_{r,r',i}^e PE_{r,r',i}WTF_{r,r',i}$	$\sum_i \tau_{r',r,i}^m WPM_{r',r,i}WTF_{r',r,i}$	$\chi_r^c k_{r,h}^h YH_r$
Investment	1						$S_{r,h}^h$
Depreciation	1						$DEPRY_r$
Trade	r'						
International trade margins	1						
Balance of payments	1						

Figure A6-2: The schematic Social Accounting Matrix, continued

		GOV	INV	DEPR	TRADE	ITT_MARG	BOP
		1	1	1	r'	1	1
Activities	i						
Produced commodities	i				$WPE_{r,r',i} WTF_{r,r',i}$	$PP_{r,i} XMG_{r,i}$	
Domestic commodities	i	$PD_{r,i} XD_{r,i,Gov}$	$PD_{r,i} XD_{r,i,Inv}$				
Imported commodities	i	$PMT_{r,i} XM_{r,i,Gov}$	$PMT_{r,i} XM_{r,i,Inv}$				
Sales tax on dom. goods	i	$\tau_{r,i,Gov}^{Ad} PD_{r,i} XD_{r,i,Gov}$	$\tau_{r,i,Inv}^{Ad} PD_{r,i} XD_{r,i,Inv}$				
Sales tax on imp. goods	i	$\tau_{r,i,Gov}^{Am} PMT_{r,i} XM_{r,i,Gov}$	$\tau_{r,i,Inv}^{Am} PMT_{r,i} XM_{r,i,Inv}$				
Value added	fp						
Value added tax	fp						
Production tax	1						
Import tariffs	r'						
Export taxes	r'						
Households	1						
Government	1						
Investment	1	S_r^g		$DEPRY$			S_r^f
Depreciation	1						
Trade	r'						$\sum_i WPE_{r,r',i} WTF_{r,r',i}$
International trade margins	1						$\sum_i PP_{r,i} XMG_{r,i}$
Balance of payments	1				$\sum_i WPM_{r',r,j} WTF_{r',r,j}$		

Annex 8: Dimensions of GTAP release 7.1

Table A8-1: Regional Concordance

1	AUS	Australia
2	NZL	New Zealand
3	XOC	Rest of Oceania <i>American Samoa (asm), Cook Islands (cok), Fiji (fji), French Polynesia (pyf), Guam (gum), Kiribati (kir), Marshall Islands (mhl), Federated States of Micronesia (fsm), Nauru (nau), New Caledonia (ncl), Norfolk Island (nfk), Northern Mariana Islands (mnp), Niue (niu), Palau (plw), Papua New Guinea (png), Samoa (wsm), Solomon Islands (slb), Tokelau (tkl), Tonga (ton), Tuvalu (tuv), Vanuatu (vut), Wallis and Futura Islands (wlf)</i>
4	CHN	China
5	HKG	Hong Kong (China)
6	JPN	Japan
7	KOR	Republic of Korea
8	TWN	Taiwan (China)
9	XEA	Rest of East Asia <i>Macao (mac), Mongolia (mng), North Korea (prk)</i>
10	KHM	Cambodia
11	IDN	Indonesia
12	LAO	Lao, PDR
13	MYS	Malaysia
14	PHL	Philippines
15	SGP	Singapore
16	THA	Thailand
17	VNM	Vietnam
18	XSE	Rest of Southeast Asia <i>Brunei Darussalam (brn), Myanmar (mmr), Timor-Leste (tmp)</i>
19	BGD	Bangladesh
20	IND	India
21	LKA	Sri Lanka
22	PAK	Pakistan
23	XSA	Rest of South Asia <i>Afghanistan (afg), Bhutan (btn), Maldives (mdv), Nepal (npl)</i>
24	CAN	Canada
25	USA	United States
26	MEX	Mexico
27	XNA	Rest of North America <i>Bermuda (bmu), Greenland (grl), Saint Pierre et Miquelon (spm)</i>
28	ARG	Argentina
29	BOL	Bolivia
30	BRA	Brazil
31	CHL	Chile
32	COL	Colombia
33	ECU	Ecuador
34	PRY	Paraguay
35	PER	Peru
36	URY	Uruguay
37	VEN	Venezuela, Republica Bolivariana de
38	XSM	Rest of South America <i>Falkland Islands (flk), French Guiana (guf), Guyana (guy), Suriname (sur)</i>
39	CRI	Costa Rica
40	GTM	Guatemala
41	NIC	Nicaragua
42	PAN	Panama
43	XCA	Rest of Central America <i>Belize (blz), El Salvador (slv), Honduras (hnd)</i>
44	XCB	Caribbean <i>Anguilla (aia), Antigua & Barbuda (atg), Aruba (abw), Bahamas (bhs), Barbados (brb), Cayman Islands (cym), Cuba (cub), Dominica (dma), Dominican Republic (dom), Grenada (grd), Guadeloupe (glp), Haiti (hti), Jamaica (jam), Martinique (mtq),</i>

Montserrat (msr), Netherlands Antilles (ant), Puerto Rico (pri), Saint. Kitts & Nevis (kna), Saint Lucia (lca), Saint. Vincent and the Grenadines (vct), Trinidad & Tobago (tto), Turks and Caicos Islands (tca), British Virgin Islands(vgb), United States Virgin Islands (vir)

45	AUT	Austria
46	BEL	Belgium
47	BGR	Bulgaria
48	CYP	Cyprus
49	CZE	Czech Republic
50	DNK	Denmark
51	EST	Estonia
52	FIN	Finland
53	FRA	France
54	DEU	Germany
55	GRC	Greece
56	HUN	Hungary
57	IRL	Ireland
58	ITA	Italy
59	LVA	Latvia
60	LTU	Lithuania
61	LUX	Luxembourg
62	MLT	Malta
63	NLD	Netherlands
64	POL	Poland
65	PRT	Portugal
66	ROU	Romania
67	SVK	Slovakia
68	SVN	Slovenia
69	ESP	Spain
70	SWE	Sweden
71	GBR	United Kingdom
72	NOR	Norway
73	CHE	Switzerland
74	XEF	Rest of European Free Trade Area (EFTA) <i>Iceland (isl), Liechtenstein (lei)</i>
75	ALB	Albania
76	BLR	Belarus
77	HRV	Croatia
78	RUS	Russian Federation
79	UKR	Ukraine
80	XEE	Rest of Eastern Europe <i>Moldova (mda)</i>
81	XER	Rest of Europe <i>Andorra (and), , Bosnia and Herzegovina (bih), Faroe Islands (fro), Gibraltar (gib), Macedonia (mkd), former Yugoslav Republic of), Monaco (mco), San Marino (smr), Serbia and Montenegro (scg)</i>
82	KAZ	Kazakhstan
83	KGZ	Kyrgyz Republic
84	XSU	Rest of Former Soviet Union <i>Tajikistan (tjk), Turkmenistan (tkm), Uzbekistan (uzb)</i>
85	ARM	Armenia
86	AZE	Azerbaijan
87	GEO	Georgia
88	IRN	Iran
89	TUR	Turkey
90	XWS	Rest of Western Asia <i>Bahrain (bhr), Iraq (irq), Israel (isr), Jordan (jor), Kuwait (kwt), Lebanon (lbn), West Bank and Gaza (pse), Oman (omn), Qatar (qat), Saudi Arabia (sau), Syrian Arab Republic (syr), United Arab Emirates (are), Republic of Yemen (yem)</i>
91	EGY	Egypt
92	MAR	Morocco
93	TUN	Tunisia
94	XNF	Rest of North Africa <i>Algeria (dza), Libyan Arab Jamahiriya (lby)</i>
95	NGA	Nigeria
96	SEN	Senegal

97	XWF	Rest of Western Africa <i>Benin (ben), Burkina Faso (bfa), Cape Verde (cpv), Côte d'Ivoire (civ), Gambia, The (gmb), Ghana (gha), Guinea (gin), Guinea-Bissau (gnb), Liberia (lbr), Mali (mli), Mauritania (mrt), Niger (ner), Saint Helena (shn), Sierra Leone (sle), Togo (tgo)</i>
98	XCF	Central Africa <i>Cameroon (cmr), Central African Republic (caf), Chad (tcd), Congo (cog), Equatorial Guinea (gnq), Gabon (gab), Sao Tome & Principe (stp)</i>
99	XAC	South-Central Africa <i>Angola (ago), Democratic Republic of the Congo (cod, formerly Zaïre)</i>
100	ETH	Ethiopia
101	MDG	Madagascar
102	MWI	Malawi
103	MUS	Mauritius
104	MOZ	Mozambique
105	TZA	Tanzania
106	UGA	Uganda
107	ZMB	Zambia
108	ZWE	Zimbabwe
109	XEC	Rest of Eastern Africa <i>Burundi (bdi), Comoros (com), Djibouti (dji), Eritrea (eri), Kenya (ken), Mayotte (myt), Réunion (reu), Rwanda (rwa), Seychelles Islands (syc), Somalia (som), Sudan (sdn)</i>
110	BWA	Botswana
111	ZAF	South Africa
112	XSS	Rest of South African Customs Union <i>Lesotho (lso), Namibia (nam), Swaziland (swz)</i>

Table A8-2: Sectoral Concordance

1	PDR	Paddy rice
2	WHT	Wheat
3	GRO	Cereal grains, n.e.s.
4	V_F	Vegetables and fruits
5	OSD	Oil seeds
6	C_B	Sugar cane and sugar beet
7	PFB	Plant-based fibers
8	OCR	Crops, n.e.s.
9	CTL	Bovine cattle, sheep and goats, horses
10	OAP	Animal products n.e.s.
11	RMK	Raw milk
12	WOL	Wool, silk-worm cocoons
13	FRS	Forestry
14	FSH	Fishing
15	COA	Coal
16	OIL	Oil
17	GAS	Gas
18	OMN	Minerals n.e.s.
19	CMT	Bovine cattle, sheep and goat, horse meat products
20	OMT	Meat products n.e.s.
21	VOL	Vegetable oils and fats
22	MIL	Dairy products
23	PCR	Processed rice
24	SGR	Sugar
25	OFD	Food products n.e.s.
26	B_T	Beverages and tobacco products
27	TEX	Textiles
28	WAP	Wearing apparel
29	LEA	Leather products
30	LUM	Wood products
31	PPP	Paper products, publishing
32	P_C	Petroleum, coal products
33	CRP	Chemical, rubber, plastic products
34	NMM	Mineral products n.e.s.
35	I_S	Ferrous metals
36	NFM	Metals n.e.s.
37	FMP	Metal products
38	MVH	Motor vehicles and parts
39	OTN	Transport equipment n.e.s.
40	ELE	Electronic equipment
41	OME	Machinery and equipment n.e.s.
42	OMF	Manufactures n.e.s.
43	ELY	Electricity
44	GDT	Gas manufacture, distribution
45	WTR	Water
46	CNS	Construction
47	TRD	Trade
48	OTP	Transport n.e.s.
49	WTP	Sea transport
50	ATP	Air transport
51	CMN	Communication
52	OFI	Financial services n.e.s.
53	ISR	Insurance
54	OBS	Business services n.e.s.
55	ROS	Recreation and other services
56	OSG	Public administration and defense, education, health services
57	DWE	Dwellings

Figures

Figure 1: Production structure nesting

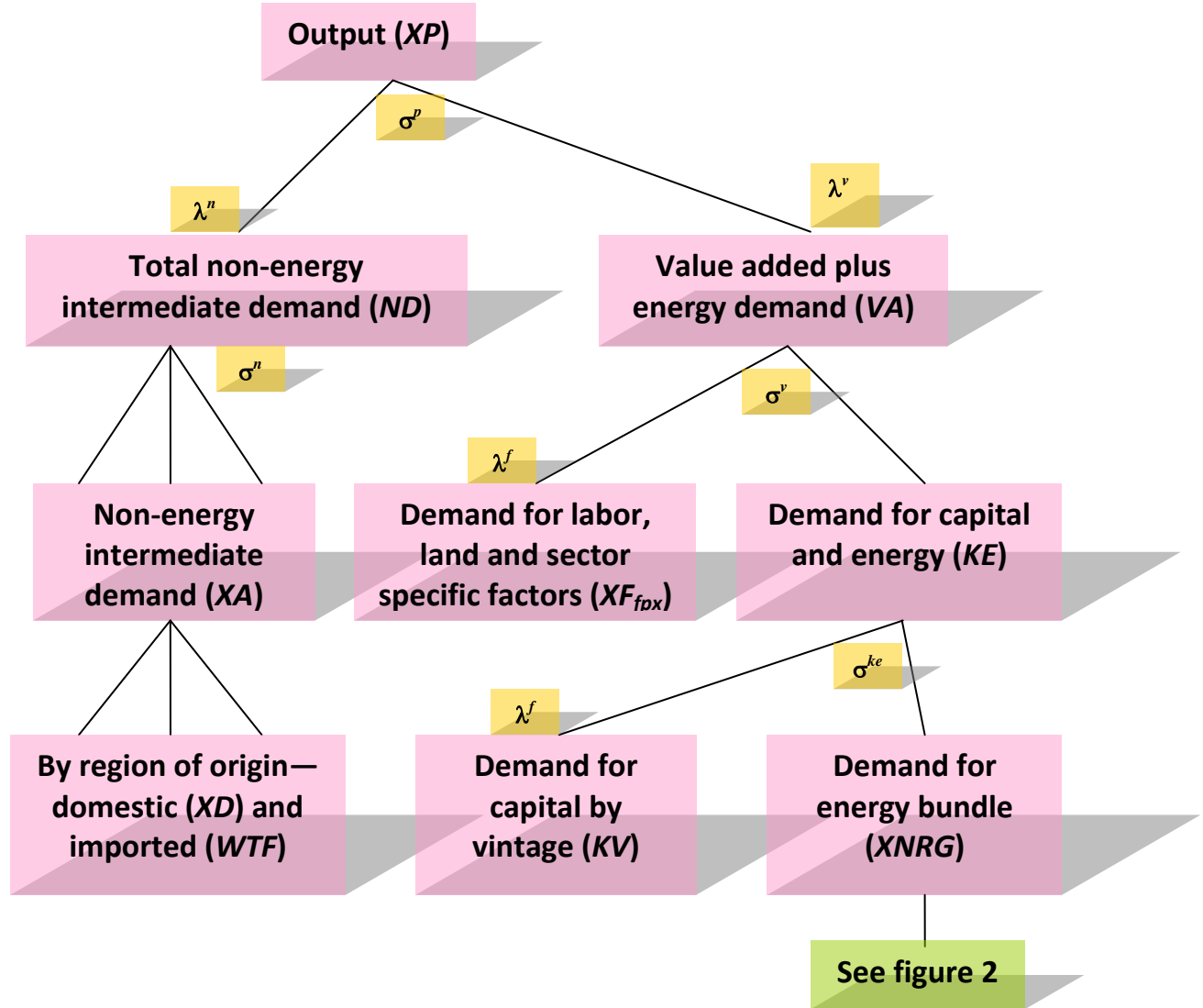


Figure 2: Energy nesting

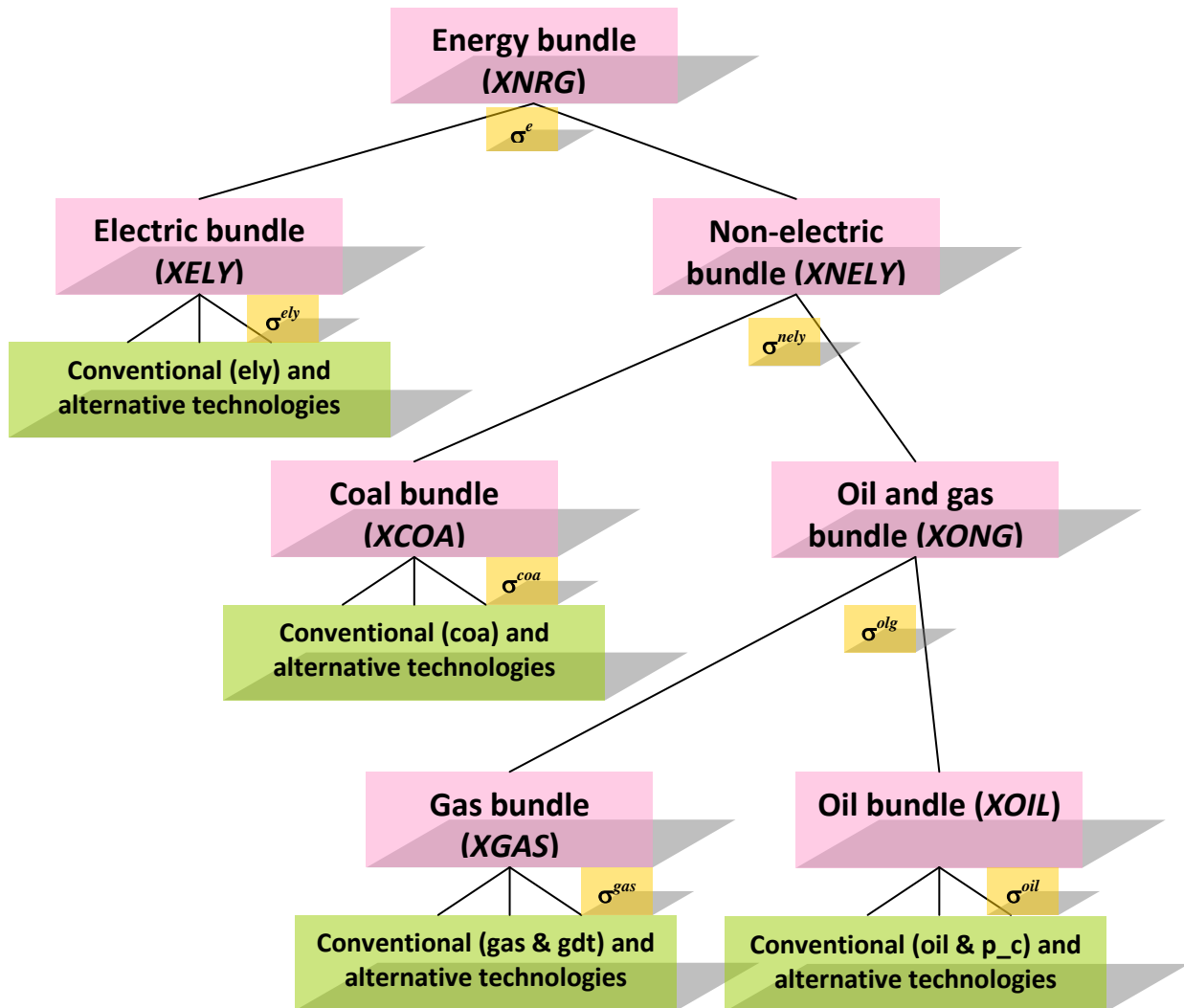


Figure 3: Output, supply and trade

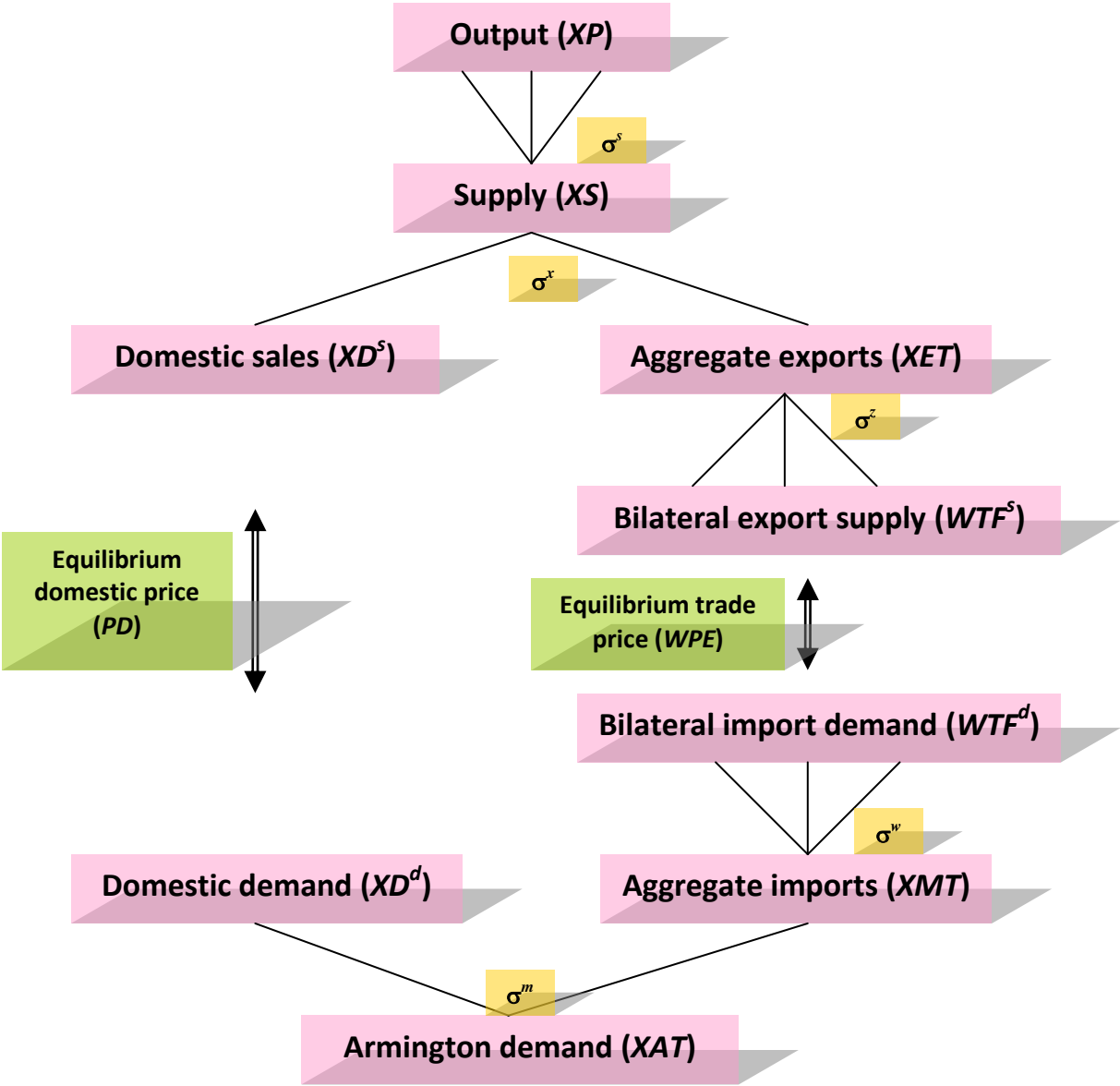


Figure 4: Domestic demand nesting

The allocation of national income across expenditure categories is determined by the closure rules. By default all net factor income accrues to households. Households allocate their disposable income between savings and expenditures on goods and services. The savings function depends on a number of factors including demographic variables. Public expenditures are fixed as a share of nominal GDP. Investment expenditures are constrained by available savings—household, public (normally set at zero) and foreign. The default household demand specification is the CDE, but the model includes the ELES and AIDADS as well.

